COLLEGE ALGEBRA

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College Algebra

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Preface

This book contains those topics in algebra that a student should master before he takes a first course in calculus. The book is intended for use in a three-hour, one-semester or a four-hour, one-quarter course. The prerequisites for college algebra are (a) an equivalent of one year of high school geometry and (b) one year of high school algebra or the course in beginning algebra offered in some junior colleges.

I have made every effort to write a book that the students can read advantageously on their own. Many examples stressing the computational aspects are included in each section. The book contains numerous problems ranging from direct substitutions to the more challenging. The problems are an integral part of the text, and the student should be reminded that it is by doing mathematics that one learns mathematics. Review questions are included at the end of each chapter. Answers to selected problems appear at the back of the book.

I am grateful for the suggestions for improving the manuscript that were made by the reviewers, Professors James Stakkestad, Robert McGuigan, Dorothy Shrader, and Fred Toxopeus. Special thanks are due my colleagues, Professors Joseph Liang, who read every line of the manuscript and made several suggestions, and James Gard, who worked all the problems in the book. I am also grateful to Mrs. Sybil French and Ms. Janelle Fortson for the outstanding job they performed in typing the manuscript. Finally, I appreciate the efforts of Mr. Everett Smethurst, the mathematics editor of Macmillan.

Tampa, Florida *J. S. R*

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Basic Concepts of Algebra

1

Our object in this chapter is to present material basic to the study of algebra. We begin with the terminology of sets and then discuss the properties of real numbers. We also discuss the important concepts of inequalities and absolute value. Other topics covered in this chapter are exponents, radicals, polynomials, and rational expressions.

Sets 1.1

The theory of sets plays a central role in the foundations of modern mathematics. Although we shall not plunge deeply into the formal theory of sets, a few of its basic concepts will be presented to give precision to our ideas.

A *set* is any well-defined collection of objects. Each object in a set is called an *element* of the set. The main characteristic of a set is that it is well defined. That is, given any particular object, it must be clear whether the object is an element of the set or not. For example, the collection of all intelligent students at your university is considered ill defined, because it is unlikely that we all can agree on its members. On the other hand, the collection of all those students who maintained a grade-point average of 3.5 or higher at your university during the current academic year is well defined.

We shall denote sets by uppercase letters, such as $A, B, C, \ldots$. We write in symbols

$$a \in A, \qquad b \notin A$$

(read "a is an element of A, b is not an element of A) to indicate that a is in the set A and b is not.

A set may have only one element, such as the set consisting of the current president of the United States of America; several elements, such as the set whose elements are the states of the U.S.A.; an infinite number of elements, such as the set of counting numbers $1, 2, 3, \ldots$; or no elements at all, such as the set of all women presidents of the U.S.A. as of 1975.

The set that has no elements is called the *empty set* or void set and is denoted by $\varnothing$ or $\{\quad\}$. The reader should note that $\{0\}$ is not the empty set. This set has one element, zero.

Set descriptions are usually included within braces. If the set does not have "too" many elements, we can tabulate all its elements. For example, $A = \{a, b, c\}$ denotes the set A whose elements are a, b, and c and no others.

Besides tabulation, a set is often specified by stating a condition or property which is satisfied by the elements of the set and is written in the form

$$A = \{x \mid x \text{ has property } P\},$$

which is read "A is the set of all elements x such that x has property P." For example, $E = \{x \mid x \text{ is an even counting number}\}$ is read "E is the set of all x such that x is an even counting number." We note that $2 \in E$, $4 \in E$, $6 \in E$, and so on; and $1 \notin E$, $3 \notin E$.

If every element of a set A is also an element of a set B, then A is called a *subset* of B and we write $A \subseteq B$ or $B \supseteq A$, which may also be read as A is contained in B or B contains A. It is obvious that $A \subseteq A$. The symbol $A \nsubseteq B$ means that A is not a subset of B.

DEFINITION 1. Two sets A and B are said to be *equal*, written $A = B$, if $A \subseteq B$ and $B \subseteq A$.

Thus the two sets A and B are equal if they contain precisely the same elements. The symbol $A \neq B$ means that A is not equal to B. If $A \subseteq B$ and $A \neq B$, then A is said to be a proper subset of B and is written $A \subset B$. In this situation there is at least one element of B that is not in A.

EXAMPLE 1. List all subsets of the set

$$A = \{a, b, c\}.$$

Solution. There are eight subsets of A. They are

$$\{a\},\ \{b\},\ \{c\},\ \{a, b\},\ \{b, c\},\ \{a, c\},\ \{a, b, c\},\ \varnothing. \quad ∎$$

We note that the set consisting of a single element a is written as $\{a\}$. It is important to note the difference between a and $\{a\}$. We also note that the empty set $\varnothing$ is not written $\{\varnothing\}$. In fact, $\varnothing \neq \{\varnothing\}$, since the empty set $\varnothing$ has no elements, whereas the set $\{\varnothing\}$ has one element, $\varnothing$. The set $\varnothing$ is a subset of every set.

When discussing sets, we always assume that they are subsets of a larger set U, called the *universe of discourse* or the *universal set*. Ideas involving the universal set, its subsets, and certain operations on sets can be visualized by using plane geometric figures called *Venn diagrams*. Figure 1 shows such a diagram. Here all points of the rectangle and its interior represent the universal

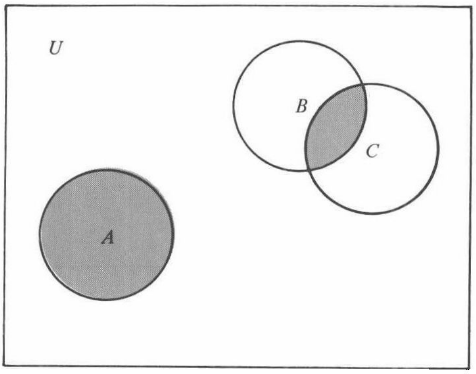

FIGURE 1

set U. It also shows some of the subsets of U, denoted by circles and their interiors.

We now define some operations on the subsets of a given universal set U.

DEFINITION 2. The *union* of two subsets A and B of U is the set of all elements of U that belong to either A or B or to both.

The symbol $\cup$ is used to denote the union of sets. Thus $A \cup B$ (read "A union B") is the set of all elements that are either in A or in B or in both. Figure 2 is a Venn diagram in which the shaded region depicts $A \cup B$.

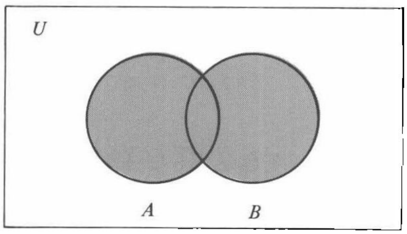

FIGURE 2

EXAMPLE 2. Let $A = \{2, 3, 4, 6, 8\}$ and $B = \{1, 3, 5, 6, 7\}$. Find $A \cup B$.

Solution

$$A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}.$$ ∎

Observe that the elements 3 and 6 appear in both sets, and yet when we form the union, we list each of these elements only once. (Why?)

A second operation of interest is given in

DEFINITION 3. The *intersection* of two subsets A and B of U is the set of all elements of U that belong to both A and B.

The symbol $\cap$ is used to denote the intersection of sets. Thus $A \cap B$ (read "A intersection B") denotes the set of all elements that are in both A and B.

In the Venn diagram of Figure 3, $A \cap B$ is represented by the shaded region.

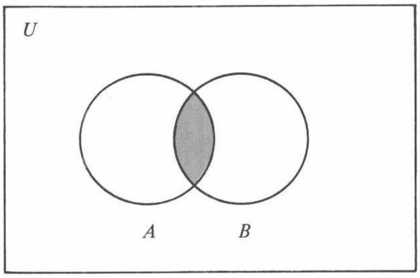

FIGURE 3

EXAMPLE 3. For the sets A and B of Example 2, find $A \cap B$.

Solution

$$A \cap B = \{3, 6\}.$$ ∎

If $A \cap B = \varnothing$, that is, if A and B have no elements in common, then A and B are said to be *disjoint* sets.

The operations of union and intersection can be performed on three or more sets by grouping them in pairs.

EXAMPLE 4. Let A and B be sets of Example 2 and let $C = \{2, 3, 9\}$. Find

(a) $(A \cup B) \cup C$ (b) $(A \cup B) \cap C$
(c) $(A \cap B) \cup C$ (d) $(A \cap B) \cap C$

Solution

(a) $$(A \cup B) \cup C = \{1, 2, 3, 4, 5, 6, 7, 8\} \cup \{2, 3, 9\} = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

(b) $$(A \cup B) \cap C = \{1, 2, 3, 4, 5, 6, 7, 8\} \cap \{2, 3, 9\} = \{2, 3\}$$

(c) $$(A \cap B) \cup C = \{3, 6\} \cup \{2, 3, 9\} = \{2, 3, 6, 9\}$$

(d) $$(A \cap B) \cap C = \{3, 6\} \cap \{2, 3, 9\} = \{3\}$$ ∎

EXERCISE 1

1. Write each set notation in words.
   (a) $A \subset B$ (b) $y \in A$ (c) $P \not\subseteq Q$ (d) $x \notin A$ (e) $\varnothing$
   (f) $A \cup B$ (g) $A \cap B$ (h) $A \subseteq B$ (i) $\{0\}$
2. Let $A = \{1, 3, 5, 7, 8\}$ and $B = \{2, 3, 5, 6, 8\}$. Insert the correct symbol ($\in$ or $\notin$) in the following blanks.
   2 ______ A 5 ______ A 6 ______ A 7 ______ A
   2 ______ B 4 ______ B 6 ______ B 7 ______ B.

3. Which statements are true and which are false?
 (a) $2 \in \{1, 2, 3\}$ (b) $\{2\} \in \{1, 2, 3\}$
 (c) $2 \subset \{1, 2, 3\}$ (d) $\{2\} \subset \{1, 2, 3\}$
 (e) $\{2\} \subseteq \{1, 2, 3\}$ (f) $\{2\} \in \{1, \{2\}\}$
 (g) $\{1, 2, 3\} = \{2, 1, 3\}$ (h) $\{1, 2\} = \{1, 1, 2, 2, 1, 1\}$
 (i) $\{a\} = \{\{a\}\}$ (j) $\{a\} \in \{a, \{\{a\}\}\}$
 (k) $\{2, 3\} \subseteq \{2, 5, 7\}$ (l) $\varnothing = \{0\}$
4. Give, if possible, a property that describes each set.
 (a) $\{1, 2, 3, 4, 5, 6\}$ (b) $\{1, 3, 5, 7, 9\}$
 (c) $\{2, 4, 6, 8\}$ (d) $\{2, 3, 5, 7, 11, 13, 17, 19\}$
5. Tabulate the elements of each set.
 (a) $\{x \mid x$ is a counting number less than $13\}$
 (b) $\{x \mid x$ is an even counting number less than $21\}$
 (c) $\{x \mid x$ is a month of the year$\}$
 (d) $\{x \mid x$ is a letter of the word "parallel"$\}$
6. Which set is the empty set?
 (a) $A = \{x \mid x$ is a letter before a in the alphabet$\}$
 (b) $B = \{x \mid x$ is a counting number, $x^2 = 9$ and $2x = 4\}$
 (c) $C = \{x \mid x \neq x\}$
 (d) $D = \{x \mid x + 3 = 3\}$
7. Let $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8\}$, and $C = \{3, 4, 5, 6\}$. Find
 (a) $A \cup B$ (b) $A \cup C$ (c) $B \cup C$
 (d) $B \cup B$ (e) $(A \cup B) \cup C$ (f) $A \cup (B \cup C)$
 (g) $A \cap B$ (h) $A \cap C$ (i) $B \cap C$
 (j) $A \cap A$ (k) $(A \cap B) \cap C$ (l) $A \cap (B \cap C)$
 (m) $A \cap (B \cup C)$ (n) $A \cup (B \cap C)$ (o) $(A \cap B) \cup (A \cap C)$
8. Draw a Venn diagram for each set.
 (a) $(A \cup B) \cup C$ (b) $(A \cap B) \cap C$
 (c) $A \cap (B \cup C)$ (d) $A \cup (B \cap C)$
9. Are the following sets equal?

$$A = \{x \mid x \text{ is a letter of the word "follow"}\}$$

$$B = \{x \mid x \text{ is a letter of the word "wolf"}\}$$

$$C = \{x \mid x \text{ is a letter of the word "flow"}\}.$$

10. If a set A has 32 elements and a set B has 42 elements and $A \cup B$ contains 62 elements, find the number of elements in $A \cap B$.
11. List all the subsets of each set. Indicate which of the subsets are proper.
 (a) $\{3\}$ (b) $\{2, 3\}$ (c) $\{2, 3, 4\}$ (d) $\{1, 2, 3, 4\}$
12. If a set A has n elements, how many different subsets of the set A do we have?

The Real Numbers 1.2

One of the most important sets in mathematics is the set R of real numbers. We shall not give a detailed development of the real number system, but briefly give some of the axioms and properties of R and review the terminology associated with the real numbers.

The two basic operations of the real number system are addition, denoted by "+," and multiplication, denoted by "·" or juxtaposition.

A1. CLOSURE PROPERTY. The set R is closed with respect to addition and multiplication. This means that if x and y are real numbers, then so are the unique numbers $x + y$ and $x \cdot y$ (also denoted by xy). The numbers $x + y$ and xy are called the *sum* and *product*, respectively, of x and y.

A2. COMMUTATIVE LAWS OF ADDITION AND MULTIPLICATION. For every pair x, y in R,

$$x + y = y + x$$
$$x \cdot y = y \cdot x.$$

A3. ASSOCIATIVE LAWS FOR ADDITION AND MULTIPLICATION. For every x, y, z in R,

$$(x + y) + z = x + (y + z)$$
$$(x \cdot y) \cdot z = x \cdot (y \cdot z).$$

A4. DISTRIBUTIVE LAWS. For every x, y, z in R,

$$(x + y) \cdot z = x \cdot z + y \cdot z$$
$$x \cdot (y + z) = x \cdot y + x \cdot z.$$

A5. IDENTITY ELEMENTS. There exist real numbers zero (denoted by 0) and one (denoted by 1) such that for every $x \in R$,

$$x + 0 = x = 0 + x$$
$$x \cdot 1 = x = 1 \cdot x.$$

A6. INVERSE ELEMENTS. For each $x \in R$, there is an element called the *negative* of x in R, denoted by $-x$, such that

$$x + (-x) = 0 = (-x) + x.$$

For each real number $x \neq 0$, there is an element called the *reciprocal* of x in R, denoted by $\frac{1}{x}$ or x^{-1}, such that

$$x \cdot \frac{1}{x} = 1 = \frac{1}{x} \cdot x.$$

Properties A1–A6 are assumptions that we make about the real number system, and they are not to be proved.

Since by A3,

$$(x + y) + z = x + (y + z),$$

it is common practice to write $x + y + z$ to represent the real number $(x + y) + z$ or $x + (y + z)$. Similarly, we write xyz in place of $(xy)z$ or $x(yz)$.

Many other properties of the real numbers can be proved by using axioms A1–A6. We list here some of the important ones:

Let x, y, and z be real numbers:

(1) If $x + z = y + z$, then $x = y$.

(2) If $x \cdot z = y \cdot z$ and $z \neq 0$, then $x = y$.

(3) If $x \cdot y = 0$, then either $x = 0$ or $y = 0$.

Property (1) is called the *cancellation law for addition,* (2) is called the *cancellation law for multiplication,* and (3) is called the *principle of zero products.*

For $a, b \in R$:

(4) $$a \cdot 0 = 0.$$

(5) $$(-1)a = -a.$$

(6) $$-(-a) = a.$$

(7) $$-(a + b) = (-a) + (-b).$$

(8) $$(-a)b = a(-b) = -ab.$$

(9) $$(-a)(-b) = ab.$$

(10) $$\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b},\ b \neq 0.$$

(11) $$\frac{-a}{-b} = \frac{a}{b},\ b \neq 0.$$

The other two operations of arithmetic, subtraction and division, may be defined in terms of addition and multiplication, respectively.

DEFINITION 4. The operation of *subtraction,* denoted by "$-$," is defined by

$$a - b = a + (-b),$$

and the operation of *division,* denoted by "$\div$," is defined by

$$a \div b = a \cdot \frac{1}{b}, \qquad b \neq 0.$$

The symbol $a \div b$ is often replaced by the symbol $\frac{a}{b}$ or a/b and we refer to it as the *quotient* of a by b, or the *fraction* a over b. The number a is called the *numerator* and b is called the *denominator* of the fraction.

Note that $a \div 0$ or $\frac{a}{0}$ is not defined. Thus division by zero is not permissible.

The following rules for operation for fractions can be established. We assume that all denominators are nonzero real numbers.

(12) $$b\left(\frac{a}{b}\right) = a.$$

EXAMPLE: $5\left(\frac{2}{5}\right) = 2.$

(13) $$\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab}.$$

EXAMPLE: $\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}.$

(14) $$\frac{1}{1/a} = a.$$

EXAMPLE: $\frac{1}{\frac{1}{3}} = 3.$

(15) $$\frac{a}{b} = \frac{c}{d} \text{ if and only if } ad = bc.$$

EXAMPLE: If $\frac{2}{4} = \frac{3}{6}$, then $2 \cdot 6 = 3 \cdot 4$, and conversely.

(16) $$\frac{ad}{bd} = \frac{a}{b}.$$

EXAMPLE: $\frac{35}{42} = \frac{5 \cdot 7}{6 \cdot 7} = \frac{5}{6}.$

(17) $$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

EXAMPLE: $\frac{2}{3} \cdot \frac{4}{7} = \frac{8}{21}.$

(18) $$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}.$$

EXAMPLE: $\frac{2}{3} + \frac{5}{3} = \frac{7}{3}.$

(19) $$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}.$$

EXAMPLE: $\frac{2}{3} + \frac{3}{4} = \frac{2 \cdot 4 + 3 \cdot 3}{3 \cdot 4} = \frac{8+9}{12} = \frac{17}{12}.$

(20) $$\frac{a/b}{c/d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}.$$

EXAMPLE: $\frac{\frac{2}{3}}{\frac{3}{4}} = \frac{2 \cdot 4}{3 \cdot 3} = \frac{8}{9}.$

By repeatedly adding 1, we generate the set of *positive integers*

$$N = \{1, 2, 3, \ldots\}.$$

This set is also called the set of *natural numbers* or *counting numbers*. The negatives of these positive integers form a set of *negative integers*

$$\bar{N} = \{-1, -2, -3, \ldots\}.$$

The set

$$\begin{aligned} Z &= \bar{N} \cup N \cup \{0\} \\ &= \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\} \end{aligned}$$

is called the set of *integers*.

A number x is said to be *rational* if it can be expressed in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$. For example, 2.5, $-\frac{5}{7}$, and 5 are rational numbers, since they can be expressed as $\frac{5}{2}$, $\frac{-5}{7}$, and $\frac{5}{1}$, respectively. The set of all rational numbers is denoted by Q. Thus

$$Q = \left\{x \mid x = \frac{a}{b},\ a, b \in Z, \text{ and } b \neq 0\right\}.$$

If a and b do not have a common factor other than 1, the rational number $\frac{a}{b}$ is said to be in its "lowest terms." For example, $\frac{8}{12}$ can be reduced in lowest terms to $\frac{2}{3}$ by multiplying numerator and denominator of $\frac{8}{12}$ by $\frac{1}{4}$. We note that it is permissible to multiply (divide) numerator and denominator by a nonzero real number. However, if we add (or subtract) a nonzero real number to the numerator and denominator, we change the original expression: for example,

$$\frac{1}{2} \neq \frac{1+3}{2+3} = \frac{4}{5}.$$

We have the following relationship among the sets N, Z, Q, and R:

$$N \subset Z \subset Q \subset R.$$

The rational numbers can also be described by their decimal representations. For example, let us consider the specific rational number $\frac{4}{7}$. If we divide 4 by 7 (long division) we see that

$$\frac{4}{7} = 0.571428\overline{571428}\ldots,$$

where the bar identifies the block of digits that repeats infinitely often. We generalize this concept as follows. Suppose that $\frac{a}{b}$ is a rational number with $b \in N$. When we perform the division $a \div b$, we notice that in each step within the division process, the remainders must be either $0, 1, 2, \ldots, b - 1$. (Why?) Thus we continue the division process until a remainder repeats. When the remainder repeats, the digits in the quotient repeat. Consequently, every rational number can be represented by an eventually repeating decimal. The converse of this statement is also true; that is, every eventually repeating decimal represents a rational number.

EXAMPLE 1. Express the rational number

$$0.4545\overline{45}$$

as the ratio of two integers.

Solution. Let $x = 0.4545\overline{45}$. Multiplying both sides of this equation by 100, we get

$$100x = 45.45\overline{45}$$

and

$$x = 0.45\overline{45}.$$

Subtracting the corresponding sides from both equations, we have

$$99x = 45$$

or

$$x = \frac{45}{99} = \frac{5}{11}.$$ ∎

EXAMPLE 2. Express $2.1232\overline{323}$ as the ratio of two integers.

Solution. Let $x = 2.1232\overline{323}$. Multiplying by 1000 moves one of the repeating blocks of the decimals to the left of the decimal. We get

$$1000x = 2123.2\overline{323}. \tag{21}$$

We next multiply x by 10, to obtain

$$10x = 21.2\overline{323}. \tag{22}$$

Subtracting the corresponding sides of (22) from (21), we have

$$990x = 2102$$

or

$$x = \frac{2102}{990} = \frac{1051}{495}.$$

∎

Now it is easy to give an example of a decimal expression that is nonrepeating. For example, we let

$$x = 3.02002000200002\ldots,$$

where there is one more "0" after each "2" than there is before the "2." Thus the number x given above is not rational. Such numbers are called *irrational numbers*.

The number $\sqrt{2}$ is irrational, because we can show that it cannot be expressed as the ratio of two integers. Other examples of irrational numbers are $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, $2 + \sqrt{3}$, π, and $\sqrt{2} + \sqrt{5}$.

The set of real numbers, in a sense, may be considered as the set of all numbers that can be written as decimal numbers. Consequently, the set R of real numbers is the union of two disjoint sets: Q, the set of rational numbers or repeating decimals, and I, the set of irrational numbers or nonrepeating decimals. In symbols,

$$R = Q \cup I \quad \text{and} \quad Q \cap I = \varnothing.$$

We can also show that between any two real numbers there are infinitely many rational numbers and infinitely many irrational numbers. For example, between 1.1 and 1.11, we can insert the rational numbers:

1.101
1.1001
1.10001
1.100001 etc.

Also, we can insert infinitely many irrational numbers:

1.1023233233323333...
1.10023233233323333...
1.100023233233323333... etc.

EXERCISE 2

In problems 1 through 12, justify each equality by stating only one of the properties (A1–A6) of real numbers.