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A. Anile Y. Choquet-Bruhat (Eds.)

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Noto 1987

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Lectures given at the 1st 1987 Session of the
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continued on page 313

PREFACE

In the field of Relativistic Fluid Dynamics, there has been only one previous conference (the C.I.M.E. course of 1970, held in Bressanone with the late professor Cattaneo as Director) and the only other book on the subject is the excellent monograph by professor Lichnerowicz, dated 1967, entitled **Relativistic Hydrodynamics and Magnetohydrodynamics** and published by Benjamin. Therefore it is no surprise that after 17 years the proceedings of a course on this subject should amount to a rather substantial book. In 17 years the subject has developed greatly, mainly with regard to applications which previously would never have been imagined.

In particular there has been a tremendous development in the field of plasma physics (relativistic fluids are a good model for high-energy astrophysical plasmas) and nuclear physics (relativistic fluids are currently used in the analysis of the heavy ion reactions). Therefore relativistic fluid dynamics is a working tool in vastly different areas such as astrophysical plasmas and nuclear physics.

This is the explanation for the fact that, since 1970, there has been no other general course on the subject. In fact there have been sessions on relativistic fluids in conferences on plasma physics and on nuclear physics separately. However this tended to obscure the underlying mathematical structure of the subject and made more difficult to transfer results and techniques from one area to another.

Having realized this, we thought that a course on this subject could bring expertise and interest from several areas (astrophysics, plasma physics, nuclear physics, mathematical methods) and provide an appropriate arena for fruitful discussions and exchanges of ideas

The main lecture courses had the objective of introducing the most significant aspects of relativistic fluid dynamics. Their topics were: **covariant theory of conductivity in ideal fluid and solid media; covariant fluid mechanics and thermodynamics: an Introduction; hamilton techniques for relativistic fluid dynamics;** and **stability theory, relativistic plasmas.**

The lectures were delivered by leading scientists in these areas (B. Carter, W. Israel, D. Holm, H. Weitzner) and constitute an up-to-date and thorough treatment of these topics.

They were also several interesting contributions from the seminars on specialized topics. Not all of them, for reasons of space, have been included in this volume. In particular, the seminars by Dudyński and Ekiel-Jezewska, Granik, Hiscock and Lindblom, Deb Ray, Boillat, were omitted. The important topics treated by these authors are covered, however, in other publications.

About fifty people (including research students and senior scientists) participated actively in the course.

We thank all the lecturers and the participants for their invaluable contribution to the success of the course. We thank also the C.I.M.E. foundation and its Director, Professor Conti and secretary, Professor Zecca, for having sponsored the Course and for their constant help and encouragement. Thanks are also due to the City of Noto (world famous for its beautiful beaches and splendid baroque architecture) for its support of the conference and the lavish hospitality. Finally we are grateful to the local organizing committee (Dr. Muscato, Professors Miceli, Fianchino and Fortuna) for their support and dedication to the success of the meeting.

TABLE OF CONTENTS

Main lectures

Contributed papers

COVARIANT THEORY OF CONDUCTIVITY IN IDEAL FLUID OR SOLID MEDIA.

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Contents:

Abstract

After a preparatory account of the established theory of non-conducting perfect fluid media, with emphasis on the important but traditionally neglected concept of the 4-momentum 1-form associated with each chemically independent constituent, it is shown how to generalise the theory to allow for conductivity by extending the variational formalism in terms of independent displacements of the world-lines.

Attention is concentrated initially on the simplest possible conducting model, in which appart from a single conserved particle current the only other constituent is the entropy-current whose flow world-lines are displaced independently of those of the conserved particles in the variational formulation, resistive dissipation being included by allowing the variationally defined force density acting between the particle and entropy currents to be non-zero. The model so obtained is fully determined by the specification of the resistivity coefficient and the traditional thermodynamic variables of the corresponding non-conducting thermal equilibrium state if it is restricted by postulating that it satisfies a "regularity ansatz" to the effect that the separate 4-momenta associated with the (non-conserved) entropy and the (conserved) particles are respectively directed allong the corresponding flow directions. It is shown that this regularity ansatz is consistent with good hyperbolic causal behaviour, unlike a previous ansatz proposed by Landau and Lifshitz, which is interpretable as a degeneracy requirement to the effect that the separate 4-momenta have the same direction as each other, and which results in (inevitably superluminal) parabolic behaviour. Another ansatz, proposed much earlier by Eckart, is shown to be effectively equivalent to the mixed-up requirement that the 4-momentum associated with the entropy to be directed not along its own flow direction but along that of the particles, and (as recently shown by Hiscock and Lindblom) results in even worse (quasi-elliptic) behavior.

After this analysis of the simplest possible well behaved thermally conducting model, it is shown how the principles by which it was constructed can be extended to allow for multiple (including electrically charged) currents, in solid as well as fluid media.

Introduction.

On of the main objectives of this course will be to demonstrate the availability of a simple and natural way of treating thermal conductivity in relativistic hydrodynamics using an effectively unique "off the peg" model to be designated by the qualification "regular", which singles it out within a wider class of in general "anomalous" (albeit mathematically well behaved and for some purposes physically well adapted) models of a somewhat more complicated type, and which distinguishes it also from the older and better known models due to Eckart[1] and to Landau and Lifshitz [2] whose mathematical behaviour has long been known to be blatently pathological, due essentialy to their failure to make proper allowance for the inertial delay time that should normally occur between the application of any external driving force (in this case the effect of a thermal gradient) and the build up of the corresponding response (in this case a proportional heat flux)[3,4].

Following lines originally developped in the non-relativistic domain by Muller[5], a considerable body of more recent work, mainly due to Israel and Stewart[6,7,8] has shown how the causal pathology in the more primitive earlier models can be satisfactorily overcome within a larger and much more elaborate class of "second order" models containing many adjustable parameters and functions that allow a model within this class to be "tailored" to fit particular physical contexts with considerable accuracy, using as a test case the much studied example of a monoatomic Boltzman gas[9]. In many practical situations, however, the cost in time (or in money, which in numerical computing and many other contexts often amounts to the same thing) of high accuracy tailoring is effectively prohibitive. Moreover lack of detailed knowledge of the subject to be fitted may render accurate tailoring impossible in any case, even if cost is no object. (Anyone with experience of shopping for clothes as a surprise present for someone else will be familiar with this problem.) It is therefore useful to have the option of using an inexpensive "off the peg model" that is guaranteed to be intrinsically trouble free as well as being reasonably well adapted to the most commonly ocurring situations, even if it cannot claim the high accuracy (at the expense of restrictive specialisation) of more elaborate models.

The "regular" model[10,11] to be described here is intended to fulfill such a need. Like the similarly motivated but unsuccessful earlier attempts by Eckart and by Landau and Lifshitz, this regular model can be considered as a limiting special case within the more general and complicated Israel-Stewart class. The mathematical properties of this entire class of models has recently become much better understood due to the work of Hiscock and Lindblom [12,13,14] who have carried out much more thorough analyses of causality and local stability properties than were available before. In particular they have cleared up the confusion that existed in the litterature on the question of whether the newer Landau-Lifshitz model was essentially distinct from the earlier Eckart model or whether it was merely the same theory (at least modulo unimportant higher order corrections) presented in terms of a different reference system. In a recent study of the special subclass of "first order" models within the general "second order" Muller-Stewart-Israel category, Hiscock and Lindblom have shown[13] that while the Landau-Lifshitz model is a a partial differential system that (like the ancient non-relativistic Fourier heat conduction model) exhibits

parabolic (instead of causally desirable hyperbolic) behaviour as had been generally realised before, on the other hand the Eckart model is even worse (with the corollary that it is an essentially distinct theory) in that it actually displays quasi-elliptic behaviour! The regular model whose use is being advocated here has not yet been subjected to a thorough Lindblom-Hiscock type analysis, but its manner of construction ensures in advance that – subject to inequalities such as as must be imposed on the equation of state even for a simple non-conducting perfect fluid model – it will be entirely free of such flaws.

The approach that lead directly to the derivation of the regular thermal conduction model presented here differed from the the traditional approach (by which its existence had been overlooked in favour of causally unsatisfactory alternatives) in that the traditional approach was primarily based on analysis of the *stress-momentum-energy tensor*, with components $T^{\mu\nu}$ say, whereas the alternative approach, as developped in the present course, attatches greater importance to the (traditionally neglected) concept of the *momentum-energy covector* with components π_μ say instead. Except for a system consisting of very weakly coupled parts, the stress-momentum-energy tensor is in general fundamentally well defined only for the system as a whole (whence the futility of the historic Abraham-Minkowski controversy about how to distinguish its "material" and "electromagnetic" contributions in a polarised and thus elecromagnetically interacting medium). On the other hand in the kind of system to be considered here, even strongly interacting currents have corresponding *separately* well defined momentum-energy covectors.

In the simplest kind of thermally conducting model (including those of Eckart and Landau-Lifshitz type) there are just two dynamically independent current vectors, namely an (in general non-conserved) entropy current, with components s^μ say, together with a single (conserved) particle number current, with components n^μ say, and there will therfore be two distinct corresponding momentum energy covectors, with components $\pi^0{}_\mu = \Theta_\mu$, and $\pi^1{}_\mu = \chi_\mu$ say (where the choice of symbols Θ and χ is intended as a reminder of the respective relationships with *temperature* and *chemical potential* that will be described in due course). In the most general models to be described, the momenta may be independent both of each other and of the corresponding currents, but in order to obtain a simple general purpose "off the peg" model in which, appart trom the specification of a resistivity scalar, all thermodynamic function of state are determined uniquely by their analogues (as presumed to be known a priori) in thermal equilibrium, then some restrictive simplifying ansatz is required. The Landau-lifshitz type models may be accounted for in this approach as being implicitly based on an ansatz to the effect that the momentum covectors Θ_μ and χ_μ in question should not be vectorially independent, but that they should be proportional to each other (thereby determining a unit covector which turns out to be the timelike eigenvector of the full stress-momentum-energy tensor): having imposed such a degeneracy condition on the momenta, it is not surprising that one obtains the degeneracy property of parabolicity for the characteristics of the corresponding system. The Eckart type models can analogously be accounted for as being based on an ansatz that is even more obviously inappropriate, namely that the *thermal* momentum-energy covector, Θ_μ should be proportional to the covariantly modified version, n_μ of the *particle* current, whose own momentum-energy covector χ_μ is thereby forced to have the "anomalous"

property of being directed elsewhere: again, it is scarcely surprising that such a mix-up leads to the flagrantly pathological property of quasi-elliptic behaviour.

Without going to the trouble of carrying out a causality analysis, it is obvious that neither of the prescriptions just described is compatible with the elementary common sense requirement of *consistency with the weakly-coupled limit* (as exemplified by the astrophysically familiar kind of situation in which the entropy is almost entirely carried by a "black-body radiation" gas of photons and perhaps electon-positron pairs, in comparatively weak interaction with a conserved background of heavy non-relativistic particles) in which the system may be approximated by two independent simple perfect fluids in which each of the momenta will necessarily have the same direction as the covariantly modified version of *is own* corresponding current, i.e Θ_μ will be proportional to s_μ, and not to n_μ which instead must be proportional to χ_μ. The "regular" model, as reccommended for "off the peg" use, is simply based on the postulate that the foregoing property of proportionality between each momentum-energy covector and the covariant version of the *corresponding* current should be preserved even when the effective coupling is strong. Since the decoupled limit is clearly well behaved in the sense of compatibility (subject to the usual inequalities) with normal causality, this good behaviour will evidently carry over into the wider class of coupled models characterised by the same "regularity ansatz".

The development of the subject in the present course will be based on a policy of adhering as closely as possible to a variational formulation at each stage, introducing dissipative effects in terms of the variationally defined "external" forces that would be required to vanish in the conservative strictly variational case. As well as showing the appropriate way to define the momentum-energy covectors that play the key role in our discussion, the variational approach has the advantage of taking care automatically of many of the mathematical self-consistency requirements that would otherwise have to be imposed on a piecemeal basis and which would end by going most of the way towards imposition of a variational structure in any case. (Any minor residual loss of generality is to be considered as acceptable according to the spirit of this course, whose purpose is to set up the simplest workable general purpose models for a treating broad classes of physical phenomena, rather than seeking to build the most elaborate and accurate models for specialised application.) The final, and most obvious (though for our main purpose accessory) bonus of the variational approach is that in ideal limit when the relevant dissipation coefficients (in our case the one of central interest being the thermal resistivity) are set equal to zero, one obtains a conservative system with the type of special properties whose implications and systematic exploitation are described in the accompanying lecture notes of Holm. Appart from the physical distinction that we shall be essentially concerned here with the inclusion of dissipative effects, a basic mathematical distinction between the approach to be developed here and the approach developped in the accompanying course of Holm is that the latter is based the use of a "(3+1)-decomposition" with respect to some specially chosen time-cordinate that is introduced so as to allow the direct adaptation to relativistic systems of methods (of generalised Hamiltonian type) originally developped in the context of Newtonian mechanics, whereas our present approach will be based on the

contrary principle (with complementary advantages and disadvantages) of adhering to a fully covariant treatment at all stages.

As compared with the accompanying course on the full class of "second order" models by Israel, the main physical restriction that will be imposed as a simplification throughout the present course is that we shall take no account of viscous effects. Although there is no reason in principle why they should not be dealt with in within the mathematical framework of the variational approach used here, the inclusion of viscous effects will be postponed for a future occasion since it would nevertheless involve technical complications that would risk obscuring some of the very simple, but until now generally overlooked, points that I hope to put over here. This course does however go beyond the accompanying courses in a different direction by allowing for "chemical" (in the general sense, including nuclear) interactions, which were not included in the previously cited work, but which are more important than viscous effects in many astrophysical contexts, and which are comparatively simple to deal with because their description can mainly be carried out in terms of scalars, as compared with the vectors and covectors needed for describing conduction effects and the tensors needed for describing viscous effects. The final section (which is included as an optional extra) also contains as its main content a description of the way to allow for the possiblity that the thermal conductivity under consideration may be occurring in a elastic-solid (as opposed to fluid) background (as would apply in the case of a neutron star crust). Although it would be mathematically simpler, allowance for viscous stress would involve a further step away from the strictly variational structure, and its description would involve further physically independent and therfore debateable postulates. On the other hand, although the technical machinery needed for dealing with (shear dependent) elastic stress is more elaborate mathematically than would be required for the inclusion of (shear-rate dependent) viscous stress, the fact that it involves no additional mechanism of dissipation makes treatment of elastic stress particularly simple from the point of view of the amount of physical input required.

I. NON-CONDUCTING MULTICONSTITUENT FLUIDS.

1.1 Mathematical requisites: Cartan derivatives and Lie derivatives.

Before describing the first of the physical models with which we shall be concerned, we shall start by explaining some of the basic mathematical machinery and terminology that will be used throughout this course. We shall work in terms of a background manifold, $\mathcal{M}$ say, with local coordinates x^μ, $\mu = 1, ..., n$, where the dimension will of course just be $n = 4$ in the ordinary space-time applications that will be considered. Familiarity with the usual Riemannian covariant differentiation operation ∇, with local coordinate representation ∇_μ, will be taken for granted. However although such a differentiation operation is generally covariant in the sense of being defined independently of any preferred linear structure, it does depend on the specification of a fundamental (pseudo-) metric

with components $g_{\mu\rho}$ say satisfying $\nabla_\nu g_{\mu\rho} = 0$. Since we shall find it profitable to work as far as possible with concepts and relationships that are covariant in the stronger sense of being independent even of the metric, we shall prefer, whenever it is feasable, to use the exterior differentiation scheme of Cartan which we now recapitulate briefly, both to fix the terminology and notation conventions, (which vary considerably throughout the physics litterature) and because its advantages in fluid mechanics (as opposed e.g. to electromagnetic theory), although coming to be more widely recognised (see e.g. the work of Schutz[15]), is not yet as widely known as it deserves to be.

The basic Cartan exterior calculus scheme is specialised in that it applies only to *covariant* tensors (which we shall distinguish from contravariant and mixed tensors by underlining) that are fully *antisymmetric*, i.e. to p-forms, $(p \leq n)$ as defined in terms of tensor components $\omega_{\mu_1 \ldots \mu_p}$ satisfying

$$\omega_{\mu_1 \ldots \mu_p} = \omega_{[\mu_1 \ldots \mu_p]} \tag{1.1}$$

(where square brackets denote antisymmetrised averaging), but the severity of this restriction is mitigated by the fact that such tensors, $\underline{\omega}$ are the only ones for which integration over a p-surface $\mathcal{S}$ is well defined in the absence of any previously specified (e.g. linear) structure on the manifold, since one can construct an (unambiguously additive) scalar by contracting such a p-form with the surface element p-vector (meaning a fully antisymmetric *contravariant* tensor, which we shall distinguish by an overhead arrow) with components $dS^{\mu_1 \ldots \mu_p}$ given in terms of a tangent space basis consisting of infinitesimal displacements $dx^\mu_{(1)}, dx^\mu_{(2)}, \ldots, dx^\mu_{(p)}$ by

$$d\vec{S} = d\vec{x}_{(1)} \wedge d\vec{x}_{(2)} \wedge \ldots \wedge d\vec{x}_{(p)} \tag{1.2}$$

where the (associative though not commutative) exterior product operation is defined in accordance with the normalisation convention introduced by Cartan (though not followed by all subsequent authors) by

$$(\underline{\omega} \wedge \underline{\Omega})_{\mu_1 \ldots \mu_p \mu_{p+1} \ldots \mu_{p+q}} = \frac{(p+q)!}{p!q!} \omega_{[\mu_1 \ldots \mu_p} \Omega_{\mu_{p+1} \ldots \mu_{p+q}]} \tag{1.3}$$

for any p-form $\underline{\omega}$ and q-form $\underline{\Omega}$. Using the notation $\rfloor$ for *inner* multiplication as defined by contraction with the normalisation convention

$$\vec{S} \rfloor \underline{\omega} = \frac{1}{p!} \omega_{\mu_1 \ldots \mu_p} dS^{\mu_1 \ldots \mu_p} \tag{1.4}$$

one can define the integral of $\underline{\omega}$ over $\mathcal{S}$ by a limit process as the surface elements are made infinitesimally small of the corresponding sum:

$$\int_{\mathcal{S}} d\vec{S} \rfloor \underline{\omega} = \underset{dS \to 0}{\mathcal{L}\mathrm{t}} \sum_{dS} d\vec{S} \rfloor \underline{\omega} \,. \tag{1.5}$$

In order to avoid confusion with the traditional physicist's use of the symbol "d" to indicate "infinitesimal variations" (i.e tangent space elements) as above, we shall not follow the newer mathematician's custom of using "d" as an abbreviation for the exterior differentiation operation definable in the more explicit notation as "$\partial\wedge$" , where ∂ denotes the elementary partial differentiation operation with coordinate representation given simply as

$$\partial_\mu = \frac{\partial}{\partial x^\mu} \tag{1.6}$$

Thus we distinguish between the infinitesimal variation $d\phi$ of a scalar field ϕ due to an infinitesimal displacement $d\vec{x}$ on the one hand, and the corresponding gradient 1-form which we denote by $\partial\phi$ (but which in customary mathematicians shorthand would be indiscriminately denoted by the same symbol as the image displacement $d\phi$) on the other hand, the relation between them being given by

$$d\phi = \partial\phi \rfloor d\vec{x} = (\partial\phi)\cdot d\vec{x} = (\partial_\mu\phi)dx^\mu \ . \tag{1.7}$$

where we introduce the traditional use of a simple dot, $\cdot$, to indicate contraction of just one pair of adjacent indices, as distinct from the contraction of all possible indices that is indicated by the symbol $\rfloor$ (the result being of course the same in this particular case). In this purely scalar example the antisymmetrised product symbol $\wedge$ is quite redundant. For a form $\underline{\omega}$ of higher order, $p \geq 1$, the antisymmetrisation indicated by the $\wedge$ symbol in the exterior product $\partial \wedge \underline{\omega}$ is a substantive requirement for general covariance, but for this very reason can in many (though by no means all) contexts, including the present work, be taken to be understood implicitly, without danger of ambiguity, even when the wedge symbol is tacitly dropped in the interest of brevity as we shall do from now on, writing $\partial\underline{\omega}$ for $\partial \wedge \underline{\omega}$ with coordinate components given by

$$(\partial\underline{\omega})_{\mu_1\mu_2\ldots\mu_{p+1}} = (p+1)\partial_{[\mu_1}\omega_{\mu_2\ldots\mu_{p+1}]} \tag{1.8}$$

The exterior differentiation operation as so defined has the well known cohomology property associated with the name of Poincaré, to the effect that for an arbitrary p-form $\underline{\omega}$

$$\partial\partial\underline{\omega} = 0 \tag{1.9}$$

and that at a local (but not necessarily global) level one has, conversely

$$\partial\underline{\Omega} = 0 \ \Rightarrow \ \exists\underline{\omega} \ : \ \underline{\Omega} = \partial\underline{\omega} \ . \tag{1.10}$$

One also has the associated Stoke's theorem property to the effect that the integral over a closed p-surface $\partial\Sigma$ bounding a $(p+1)$-volume Σ say will be given by

$$\oint_{\partial\Sigma} d\vec{S}\rfloor\underline{\omega} = \int_\Sigma d\vec{\Sigma}\rfloor\partial\underline{\omega} \ . \tag{1.11}$$

The development of the antisymmetric differential calculus can be taken considerably further so as to apply to contravariant tensors whenever a preferred volume measure n-form $\underline{\epsilon}$ is specified, since it may be used (even in the absence of any corresponding metric tensor) for relating p-forms to *dual* $(n-p)$-vectors and vice versa. Thus if $\vec{\beta}$ is a q-vector (i.e. an antisymmetric contravariant tensor of order q) then we can construct its dual $(n-q)$-form ${}_*\vec{\beta}$ according to the formula ${}_*\vec{\beta} = \vec{\beta} \rfloor \underline{\epsilon}$, i.e.

$$ {}_*\beta_{\mu_1 \dots \mu_{n-q}} = \frac{1}{q!} \beta^{\rho_1 \dots \rho_q} \epsilon_{\rho_1 \dots \rho_q \mu_1 \dots \mu_{n-q}} \,. \tag{1.12} $$

Using an upper star prefix for the inverse mapping from (covariant) p-forms to (contravariant) $(n-p)$-vectors, as defined by

$$ {}_*({}^*\underline{\omega}) = \underline{\omega} \,, \quad {}^*({}_*\vec{\beta}) = \vec{\beta} \tag{1.13} $$

the interior product of a p-form $\underline{\omega}$ and a q-vector $\vec{\beta}$ can be expressed (depending on whether p is larger or smaller than q) in terms of outer (Cartan) multiplication in one or other of the forms

$$ \begin{aligned} \vec{\beta} \rfloor \underline{\omega} &= {}_*(({}^*\underline{\omega}) \wedge \vec{\beta}) \quad \text{if} \quad p \geq q \,, \\ \vec{\beta} \lfloor \underline{\omega} &= {}^*(\underline{\omega} \wedge ({}_*\vec{\beta})) \quad \text{if} \quad q \geq p \end{aligned} \tag{1.14} $$

This suggests the convenience of defining the *inner* derivative, or "divergence" of a q-vector $\vec{\beta}$ to be

$$ \mathrm{div}\vec{\beta} = {}^*(\partial({}_*\vec{\beta})) \tag{1.15} $$

In order for this to be well defined the only prerequisite structure that has to be given on the manifold is the measure $\underline{\epsilon}$, the specification (by a choice of affine connection) of a general purpose covariant differentiation operation ∇ being unnecessary. However whenever a covariant differentiation operator actually is given, subject of course to consistency with the measure in the sense that $\nabla \underline{\epsilon} = 0$, the divergence operation defined by (1.15) will be expressible directly in coordinate or condensed notation as

$$ (\mathrm{div}\vec{\beta})^{\mu_1 \dots \mu_{q-1}} = \nabla_\lambda \, \beta^{\mu_1 \dots \mu_{q-1} \lambda}, \quad \mathrm{div}\vec{\beta} = (-1)^{q-1} \nabla \cdot \vec{\beta} \,. \tag{1.16} $$

We may use the generalised divergence relation defined by (1.15) to express the Stokes theorem (1.11) in the dual *Green Theorem* form commonly preferred by physicists:

$$ \oint_{\partial \Sigma} \underline{d}S \rfloor \vec{\beta} = (-1)^n \int_\Sigma \underline{d}\Sigma \rfloor \mathrm{div}\vec{\beta} \,. \tag{1.17} $$

where the abbreviation

$$ \underline{d}S = {}_*d\vec{S} \tag{1.18} $$

has been used for the dual surface element.

Another important kind of differentiation operation, which shares with exterior differentiation the property of bing well defined and generally covariant independently

of any background linear or Riemannian structure or even of any measure that may be present is *Lie differentiation* with respect to any smooth vector field $\vec{\xi}$ say, which we shall denote by the symbol $\vec{\xi}\mathcal{L}$. It is definable for any kind of field X (not just tensors, but also densities, affine connections, et cetera) that is geometric in the sense of being bijectively mappable by any non-singular differentiable automorphism $f: \ x \mapsto fx$ of the support manifold onto a well defined naturally induced retraction image fX,

$$f: \ X(fx) \mapsto fX(x) \ . \tag{1.19}$$

Letting $f(t)$ denote the one-parameter family of diffeomorphisms constructed by dragging the manifold a parameter distance t along the integral curves of

$$\frac{dx^\mu}{dt} = \xi^\mu \tag{1.20}$$

the corresponding Lie derivative is definable as

$$\vec{\xi}\mathcal{L}X = \frac{d}{dt}(f(t)X)\Big|_{t=0} \ . \tag{1.21}$$

In the case of a quantity that is *tensorial* with mixed indices $T_{\mu\ldots}^{\nu\rho\cdots}$ say, the Lie derivative is given explicitly by the general formula

$$\begin{aligned}(\vec{\xi}\mathcal{L}T)_{\mu\ldots}^{\nu\rho\cdots} &= \xi^\lambda \partial_\lambda T_{\mu\ldots}^{\nu\rho\cdots} + T_{\lambda\ldots}^{\nu\rho\cdots}\partial_\mu \xi^\lambda + \ \ldots \\ &\quad - T_{\mu\ldots}^{\lambda\rho\cdots}\partial_\lambda \xi^\nu - \ \ldots \end{aligned} \tag{1.22}$$

with an additional term for each further index, the most familiar special case being that of the Lie derivative of another vector field, $\vec{\eta}$ say for which one obtains the simple Lie commutator:

$$\begin{aligned}\vec{\xi}\mathcal{L}\vec{\eta} &= [\vec{\xi},\vec{\eta}] = -\vec{\eta}\mathcal{L}\vec{\xi} \ , \\ [\vec{\xi},\vec{\eta}]^\mu &= \xi^\lambda \partial_\lambda \eta^\mu - \eta^\lambda \partial_\lambda \xi^\mu \ . \end{aligned} \tag{1.23}$$

Another familiar special case concerns the spacetime metric $g_{\mu\rho}$ used for specifying the covariant differentiation operator ∇ by the requirement that it should give $\nabla_\lambda g_{\mu\rho} = 0$ for which one obtains

$$(\vec{\xi}\mathcal{L}g)_{\mu\rho} = 2\nabla_{(\mu}\xi_{\rho)} \tag{1.24}$$

(with the standard convention that round bracket on indices indicates symmetrised averaging over permutations) which vanishes when ξ is the generator of a one-parameter isometry group.

Of particular importance for our present purposes is the case of the "differential forms", i.e. covariant fully antisymmetric tensors to which the Cartan exterior differential calculus described above applies: for any p-form $\underset{\sim}{\omega}$, the Lie derivative is expressible concisely in the above notation scheme by Cartan's formula