

Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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S.M. Khaleelulla

Counterexamples in
Topological Vector Spaces



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PREFACE

During the last three decades much progress has been made in the field of topological vector spaces. Many generalizations have been introduced; this was, to a certain extent, due to the curiosity of studying topological vector spaces for which a known theorem of Functional analysis can be proved. To justify that a class C_1 of topological vector spaces is a proper generalization of another class C_2 of topological vector spaces, it is necessary to construct an example of a topological vector space belonging to C_1 but not to C_2 ; such an example is called a counterexample. In this book the author has attempted to present such counterexamples in topological vector spaces, ordered topological vector spaces, topological bases and topological algebras.

The author makes no claim to completeness, obviously because of the vastness of the subject. He makes no attempt to give due recognition to the authorship of most of the counterexamples presented in this book.

It is assumed that the reader is familiar with general topology. The reader may refer to B[18] for information about general topology.

To facilitate the reading of this book, some fundamental concepts in vector spaces and ordered vector spaces have been collected in the Chapter called 'Prerequisites'. Thereafter each Chapter begins with an introduction which presents the relevant definitions and statements of theorems and propositions with references where their proofs can be

found. For some counterexamples which require long and complicated proofs, only reference has been made to the literature where they are available.

The books and papers are listed separately in the bibliography at the end of the book. Any reference to a book is indicated by writing B [] and to a paper by P [] .

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