SECOND EDITION

Calculus

JAMES STEWART

To Vera, Sally, and Alan

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Cover: Illustrates the remarkable resemblance between the sound hole of a violin viewed from this angle and the elongated "S" of the integral sign, a notation introduced by Leibniz.

Preface

> A great discovery solves a great problem but there is a grain of discovery in the solution of any problem. Your problem may be modest; but if it challenges your curiosity and brings into play your inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery.
>
> *George Polya*

The art of teaching, Mark Van Doren said, is the art of assisting discovery. I have tried to write a book that assists students in discovering calculus—both for its practical power and its surprising beauty. In this edition, as in the first, I aim to convey to the students a sense of the utility of calculus and develop their technical competence, but I also strive to give them some appreciation for the intrinsic beauty of the subject. Newton undoubtedly experienced a sense of triumph when he made his great discoveries. I want students to share some of that excitement.

The emphasis is on understanding. Enough mathematical detail is presented so that the treatment is precise, but without allowing formalism to become obtrusive. The instructor can follow an appropriate course between intuition and rigor by choosing to include or exclude optional sections and proofs. Section 1.4, for example, on the precise definition of the limit is an optional section. Although a majority of theorems are proved in the text, some of the more difficult proofs are given in Appendix C.

While teaching from the first edition for three years, I (and my students) have had ideas, some major, some minor, for improving the exposition and organization and for adding new and better examples and exercises. I have also had the benefit of some valuable suggestions from colleagues, both friends and strangers, which have been incorporated into the second edition. Here is a summary of some of the principal changes:

■ At the request of several users, some of the applications of integration are introduced earlier. In fact, applications of integration now occur in two chapters. The applications, such as volume and work, that ordinarily require only basic techniques of integration are in Chapter 5. Those applications for which it is profitable to have studied further techniques (separable differential equations, arc length, surface area) are in Chapter 8, together with centers of mass, hydrostatic pressure, and a new section on applications to economics and biology (consumer's surplus, present value of an income stream, blood flow, cardiac output).

■ The chapter Limits and Rates of Change (now Chapter 1) has been substantially reorganized. In particular, the properties of limits are introduced earlier.

■ In Chapter 2 the section Rates of Change in the Natural and Social Sciences is placed earlier and there is increased emphasis on linear approximation.

■ The Midpoint Rule for approximate integration is now covered. It first occurs in Section 4.3, where the definite integral is introduced. The treatment of approximate integration in Section 7.8 is expanded to include comparison of errors for the Left Endpoint, Right Endpoint, Trapezoidal, Midpoint, and Simpson's rules.

■ In Chapter 6 the method for differentiating the exponential and logarithmic functions has been changed so that the exponential function is differentiated first.

■ In Chapter 10 there are more graphs of Taylor Series approximations, and Taylor's Formula is now proved in the text instead of in the exercises. Multiplication and division of power series are now covered.

■ Chapter 11 contains more applications of vectors in examples and exercises. Kepler's First Law is proved in the text although, as before, Laws 2 and 3 are left as exercises with hints.

■ Chapter 12 now contains many more computer graphics of surfaces and level curves, both in examples and exercises, and tree diagrams have been added to illustrate the Chain Rule. The geometric basis of Lagrange multipliers is explained.

■ A new section on surface area has been added in Chapter 13. Although parametric surfaces are still given a full treatment in Chapter 14, it is now possible to cover surface area and surface integrals nonparametrically.

■ There is a new appendix on complex numbers.

Problem-Solving Emphasis

My educational philosophy was strongly influenced by attending lectures of George Polya and Gabor Szego when I was a student at Stanford University. Both Polya and Szego consistently introduced a topic by relating it to something concrete or familiar. Wherever practical I have introduced topics with an intuitive geometrical or physical description and have attempted to tie mathematical concepts to the students' experiences.

I found Polya's lectures on problem solving highly inspirational and his books *How To Solve It, Mathematical Discovery*, and *Mathematics and Plausible Reasoning* have become the core text material for a mathematical problem-solving course that I instituted and teach at McMaster University. I have adapted these problem-solving strategies to the study of calculus both explicitly, by outlining strategies, and implicitly, by illustration and example.

Students usually have difficulties in situations where there is no single, well-defined procedure for obtaining the answer. I think nobody has improved very much on Polya's four-stage problem-solving strategy, and accordingly I have presented a version of this strategy on the front endpapers of this book together with a discussion of it and an example of its use in To the Student. I also urge my students to read Polya's *How to Solve It* for a more leisurely exposition of the principles. I often find myself nagging them into using these principles.

The classic calculus situations where problem-solving skills are especially important are related rates problems, maximum and minimum problems, integration, testing series, and solving differential problems. In these and other situations I have adapted Polya's strategies to the matter at hand. In particular three special sections are devoted to problem solving: 7.6 (Strategy for Integration), 10.7 (Strategy for Testing Series), and 15.4 (Strategy for Solving First Order Differential Equations).

Problems Plus and Applications Plus

In this edition I have added what I call *Problems Plus* after even-numbered chapters. These are problems that go beyond the usual exercises in one way or another and require a higher level of problem-solving ability. The very fact that they do not occur in the context of any particular chapter makes them a little more challenging. For instance, a problem that occurs after Chapter 10 need not have anything to do with Chapter 10. I particularly value problems in which a student has to combine methods from two or three different chapters. In recent years I have been testing these Problems Plus on my own students by putting them on assignments, tests, and exams. Because of their challenging nature I grade these problems in a different way. Here I reward a student significantly for ideas toward a solution and for recognizing which problem-solving principles are relevant. My aim is to teach my students to be unafraid to tackle a problem the likes of which they have never seen before.

A counterpart to the Problems Plus are the *Applications Plus*, which occur after odd-numbered chapters (starting with Chapter 3) and which tend to be challenging because they involve related concepts from science that are usually outside students' experiences. Again the idea is to combine concepts and techniques from different parts of the book. These problems are helpful in demonstrating the sheer variety of the applications of calculus but also in focusing the students' attention on the essential mathematical similarities in diverse situations in science. By solving a wide variety of concrete problems, I hope that they will come to appreciate the power of abstraction. I am grateful to Garret Etgen for amassing such a wide-ranging collection of applied problems.

Exercises and Examples

I believe that one of the reasons that the first edition has been successful at a range of educational institutions is that there has been such a wide range of abilities among my own calculus students and my goal has been not to lose any of them. In particular I did not want to lose the interest of my very best students and so I have sought to challenge them with stimulating exercises. There is nothing in any text that can compensate for a deficiency of good problems. I have selected my exercises from those used in 20 years of calculus classes and have expressly chosen examples for their instructional value. I have added 1000 new exercises to this edition, making a total of about 7500 exercises that range from the essential, routine ones to those that will challenge your best students. I have made a special effort to include unusual problems at both ends of the spectrum of difficulty. Many of the new exercises are thought provoking and occur toward the end of exercise sets.

Another reason for the first edition's success may be the heuristic flavor of many of the text examples. Examples can, and should, be more than exercise-solving "templates." Carefully constructed examples can be one of the most effective ways of leading students into more advanced material. Many examples herein are designed to promote careful thinking about the problems and ideas behind calculus while giving the students insight into why theorems and proofs are necessary.

Early Transcendentals Option

For the most part the order of topics presented is fairly traditional. The trend toward the early introduction of trigonometric functions is reflected in the placement of the differentiation of all six trigonometric functions in Chapter 2, before the Chain Rule. However, many instructors would also like to be able to use the other transcendental functions prior to the coverage of the definite integral. I have encouraged this alternative by making the introduction of the transcendental functions as flexible as possible. In Chapter 6 the exponential function is defined first, followed by the logarithmic function as its inverse. (Students have seen these functions introduced this way since high school.) Later I present the more elegant definition of the logarithm as an integral. This presentation allows the coverage of much of Chapter 6 before Chapters 4 and 5, if desired. To accommodate this choice of presentation, specially identified problems involving integrals of exponential and logarithmic functions are included at the end of the appropriate sections of Chapters 4 and 5. This order of presentation allows a faster-paced course to teach the transcendental functions and the definite integral in the first semester.

For instructors who would like to go even further in this direction I have prepared an alternative edition of this book, called *Calculus: Early Transcendentals Edition*, in which the exponential and logarithmic functions and their applications to exponential growth and decay (essentially the contents of Chapter 6 of the present edition) are presented in Chapter 3.

Design Pedagogy

One of the most striking changes in this edition is the four-color production. I admit that when Brooks/Cole first proposed a four-color book, I was very skeptical and advised against it. But when I actually sat down to think seriously about how four colors could enhance the book's design and pedagogy in the artwork, I gradually became fascinated with the idea. Of course, random use of color would detract from learning, but I have tried to devise consistent and functional color schemes for the art. I believe that students will find this new color scheme helpful for their learning.

From repeatedly correcting the same mistakes I have come to recognize consistent pitfalls that trap many students. I believe that it is best to alert students to them and in many cases have done so with the "caution" symbol. For example, on page 399 the symbol is used to warn the student against a common misuse of l'Hospital's Rule and on page 581 it is used to emphasize that if $\lim_{n\to\infty} a_n = 0$, then we cannot conclude that $\Sigma\, a_n$ is convergent.

The calculator symbol indicates an exercise, or group of exercises, that requires the use of calculator or computer. Although most of these exercises are designed to illustrate the power of algorithmic computation, some of them are designed to show machine limitations. See for instance Exercise 18 in Section 1.2. Appendix D, Lies My Calculator or Computer Told Me, alerts the student to many of the situations in which a calculator or computer can give unreliable answers.

Appendixes

In addition to the usual review of precalculus topics in the Review and Preview I have included an algebra review appendix, because so many of the errors made by students writing calculus exams are not errors in calculus itself, or even in precalculus material, but rather in very basic algebra. Appendix A contains a substantial review of elementary algebra and drill exercises. Similarly, a review of trigonometry appears in Appendix B, because a review thorough enough for students who really need it would be intrusive in the text per se.

Acknowledgments

It took me some eight years to write and test the first edition. I never suspected that it would take nearly two years to revise that same material. Much of that time was spent reading reasoned (but sometimes contradictory) advice from a large number of astute reviewers. I greatly appreciate the time they spent to understand my motivation for the approach taken. I have learned something from each of them.

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JAMES STEWART

To the Student

Reading a calculus textbook is different from reading a newspaper or a novel, or even a physics book. Don't be discouraged if you have to read a passage more than once in order to understand it. You should have pencil and paper at hand to make a calculation or sketch a diagram.

Some students start by trying their homework problems and only read the text if they get stuck on an exercise. I suggest that a far better plan is to read and understand a section of the text before attempting the exercises. In particular, you should study the definitions to see the exact meanings of the terms.

Part of the aim of this course is to train you to think logically. Learn to write the solutions of the exercises in a connected step-by-step fashion with explanatory words or symbols—not just a string of disconnected equations or formulas.

The answers to the odd-numbered exercises appear at the back of the book, in Appendix H. There are often several different forms in which to express an answer, so if your answer differs from mine, don't immediately assume that you are wrong. There may be an algebraic or trigonometric identity that connects the answers. For example, if the answer given in the back of the book is $\sqrt{2} - 1$ and you obtain $1/(1 + \sqrt{2})$, then you are right and rationalizing the denominator will show that the expressions are equivalent.

The symbol 🖩 means that a calculator (or computer) is required to do a calculation in an example or exercise. The symbol ■ indicates the end of a proof or an exercise. You will also encounter the symbol ⊘, which warns you against committing an error. I have placed this symbol in the margin in situations where I have observed that a large proportion of my students tend to make the same mistake.

Calculus is an exciting subject; I hope you find it both useful and interesting in its own right.

A Note on Logic

In understanding the theorems it is important to know the meaning of certain logical terms and symbols. If P and Q are mathematical statements, then $P \Rightarrow Q$ is read as "P

implies Q" and means the same as "If P is true, then Q is true." The *converse* of a theorem of the form $P \Rightarrow Q$ is the statement $Q \Rightarrow P$. (The converse of a theorem may or may not be true. For example, the converse of the statement "If it rains, then I take my umbrella" is "If I take my umbrella, then it rains.") The symbol $\Leftrightarrow$ indicates that two statements are equivalent. Thus $P \Leftrightarrow Q$ means that both $P \Rightarrow Q$ and $Q \Rightarrow P$. The phrase "if and only if" is also used in this situation. Thus "P is true if and only if Q is true" means the same as $P \Leftrightarrow Q$. The *contrapositive* of a theorem $P \Rightarrow Q$ is the statement that $\sim Q \Rightarrow \sim P$, where $\sim P$ means not P. So the contrapositive says "If Q is false, then P is false." Unlike converses, the contrapositive of a theorem is always true.

Methods of Problem Solving

The way to master calculus is to solve problems—lots of problems. The exercise sets at the end of each section begin with straightforward exercises designed to test your mastery of fundamental skills. Later exercises are less straightforward because they involve applications to science, because they involve more complex calculations, or because they make you think harder about what the concepts really mean. The problems at the end of exercise sets are often quite challenging and may require more than one attempt. The Problems Plus, which can be found after the even-numbered chapters, are also challenging. Often the challenge is to recognize that a problem requires combining methods from two or more different chapters. For instance, the Problems Plus after Chapter 8 don't necessarily relate to Chapter 8 but might require using knowledge from any of the chapters from 1 to 8.

If you have trouble solving any of the more challenging problems, you might find it useful to look at the Principles of Problem Solving that are printed on the front endpapers of this book. For a more thorough discussion of the problem-solving process I recommend that you consult the following books:

Polya, G. *How To Solve It* (2nd ed.). Princeton University Press, 1957.

Polya, G. *Mathematical Discovery*. New York: John Wiley and Sons, 1962.

Wickelgren, W. *How To Solve Problems*. San Francisco: W. H. Freeman, 1974.

Solow, D. *How To Read and Do Proofs*. New York: John Wiley and Sons, 1982.

Here is a problem that involves only precalculus mathematics:

> *Express the hypotenuse h of a right triangle in terms of its area A and its perimeter P.*

After reading the Principles of Problem Solving, try to solve this problem. Then look at the solution on the following page.

Example Express the hypotenuse h of a right triangle in terms of its area A and its perimeter P.

Understand the problem

Solution Let us first sort out the information by identifying the unknown quantity and the data:

$$\text{Unknown: } h$$

$$\text{Given quantities: } A, P$$

Draw a diagram

Connect the given with the unknown

Introduce something extra

It helps to draw a diagram and we do so in Figure 1.

In order to connect the given quantities to the unknown, we introduce two extra variables a and b, which are the lengths of the other two sides of the triangle. This enables us to express the given condition, which is that the triangle is right-angled, by the Pythagorean Theorem:

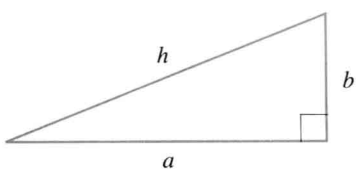

Figure 1

$$h^2 = a^2 + b^2$$

The other connections among the variables come by writing expressions for the area and perimeter:

$$A = \tfrac{1}{2}ab \qquad P = a + b + h$$

Since A and P are given, notice that we now have three equations in the three unknowns a, b, and h:

$$h^2 = a^2 + b^2 \qquad (1)$$

$$A = \tfrac{1}{2}ab \qquad (2)$$

$$P = a + b + h \qquad (3)$$

Relate to the familiar

Although we have the correct number of equations, they are not easy to solve in a straightforward fashion. But if we use the problem-solving strategy of trying to recognize something familiar, then we can solve these equations by an easier method. Look at the right sides of Equations 1, 2, and 3. Do these expressions remind you of anything familiar? Notice that they contain the ingredients of a familiar formula:

$$(a + b)^2 = a^2 + 2ab + b^2$$

Using this idea, we express $(a + b)^2$ in two ways. From Equations 1 and 2 we have

$$(a + b)^2 = (a^2 + b^2) + 2ab = h^2 + 4A$$

From Equation 3 we have

$$(a + b)^2 = (P - h)^2 = P^2 - 2Ph + h^2$$

Thus

$$h^2 + 4A = P^2 - 2Ph + h^2$$

$$2Ph = P^2 - 4A$$

$$h = \frac{P^2 - 4A}{2P}$$

This is the required expression. ■

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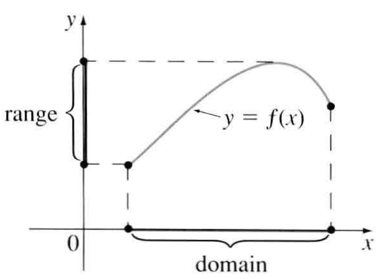

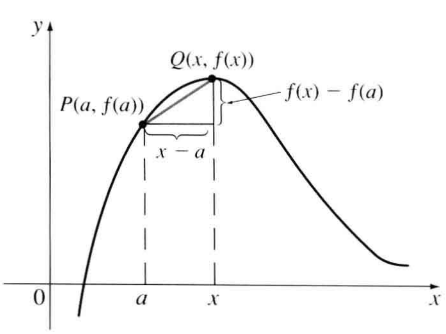

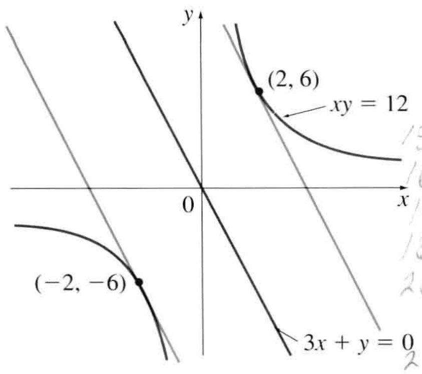

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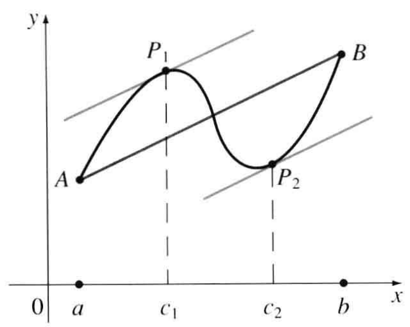

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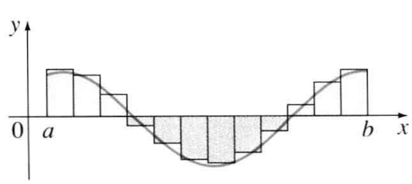

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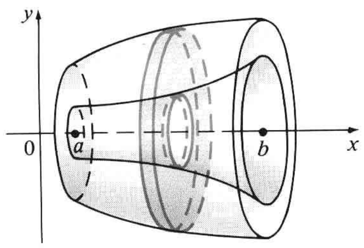
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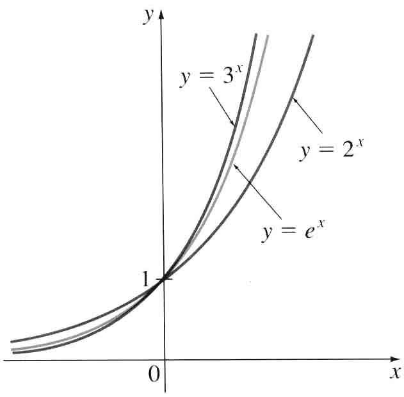
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0
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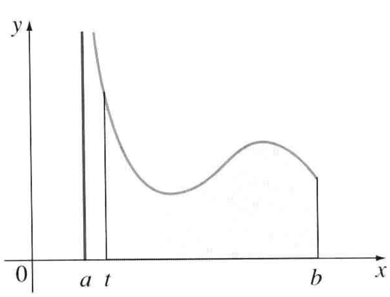
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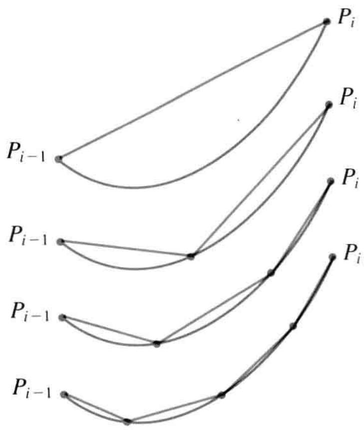
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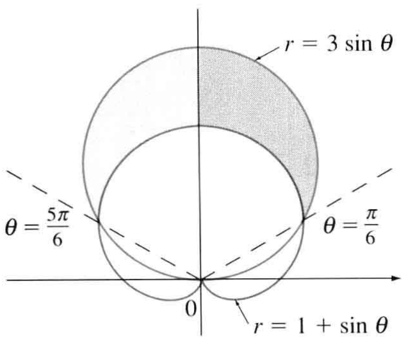
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$\theta = \frac{\pi}{6}$
0
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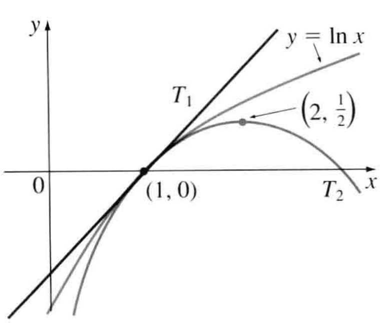
y
$y = \ln x$
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$(2, \frac{1}{2})$
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T_2
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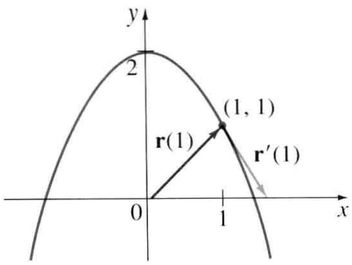
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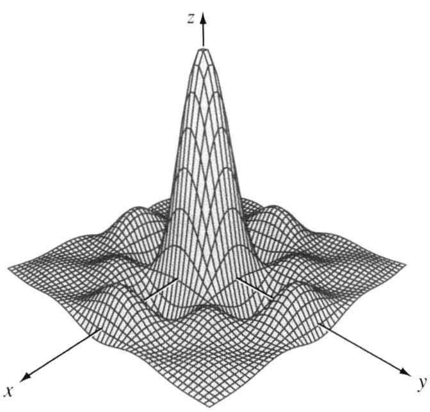
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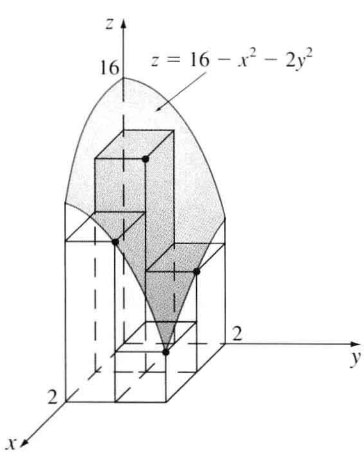
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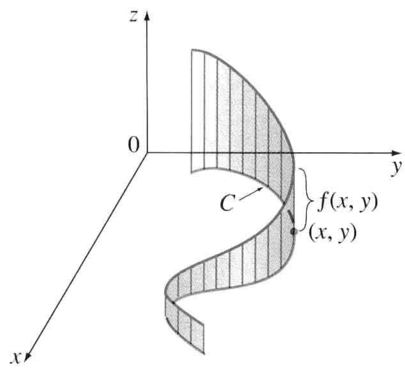

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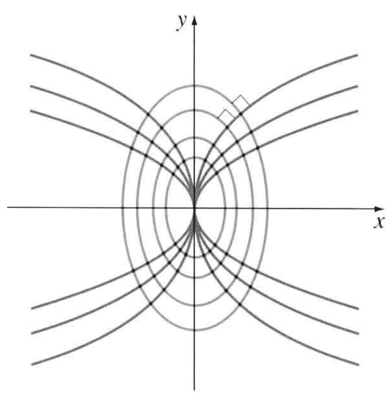

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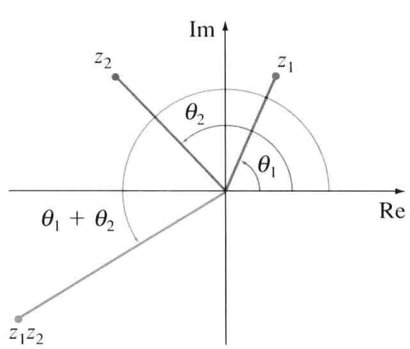

Appendixes A1

Calculus

Second Edition