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# Linear Algebra

and its Applications

*Third Edition*

David C. Lay

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# About the Author

David C. Lay holds a B.A. from Aurora University (Illinois), and an M.A. and Ph.D. from the University of California at Los Angeles. Lay has been an educator and research mathematician since 1966, mostly at the University of Maryland, College Park. He has also served as a visiting professor at the University of Amsterdam, the Free University in Amsterdam, and the University of Kaiserslautern, Germany. He has over 30 research articles published in functional analysis and linear algebra.

As a founding member of the NSF-sponsored Linear Algebra Curriculum Study Group, Lay has been a leader in the current movement to modernize the linear algebra curriculum. Lay is also a co-author of several mathematics texts, including *Introduction to Functional Analysis* with Angus E. Taylor, *Calculus and Its Applications*, with L. J. Goldstein and D. I. Schneider, and *Linear Algebra Gems—Assets for Undergraduate Mathematics*, with D. Carlson, C. R. Johnson, and A. D. Porter.

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To my wife, Lillian, and our children,  
Christina, Deborah, and Melissa,  
whose support, encouragement, and  
faithful prayers made this book possible

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# Preface

The response of students and teachers to the first two editions of *Linear Algebra and Its Applications* has been most gratifying. This Third Edition offers even more visualization of concepts, along with enhanced technology support on the web for both students and instructors. As before, the text provides a modern elementary introduction to linear algebra and a broad selection of interesting applications. The material is accessible to students with the maturity that should come from successful completion of two semesters of college-level mathematics, usually calculus.

The main goal of the text is to help students master the basic concepts and skills they will use later in their careers. The topics here follow the recommendations of the Linear Algebra Curriculum Study Group, which were based on a careful investigation of the real needs of the students and a consensus among professionals in many disciplines that use linear algebra. Hopefully, this course will be one of the most useful and interesting mathematics classes taken as an undergraduate.

## DISTINCTIVE FEATURES

### **Early Introduction of Key Concepts**

Many fundamental ideas of linear algebra are introduced within the first seven lectures, in the concrete setting of  $\mathbb{R}^n$ , and then gradually examined from different points of view. Later generalizations of these concepts appear as natural extensions of familiar ideas, visualized through the geometric intuition developed in Chapter 1. A major achievement of the text, I believe, is that the level of difficulty is fairly even throughout the course.

### **A Modern View of Matrix Multiplication**

Good notation is crucial, and the text reflects the way scientists and engineers actually use linear algebra in practice. The definitions and proofs focus on the columns of a matrix rather than on the matrix entries. A central theme is to view a matrix–vector product  $Ax$  as a linear combination of the columns of  $A$ . This modern approach simplifies many arguments, and it ties vector space ideas into the study of linear systems.

### **Linear Transformations**

Linear transformations form a “thread” that is woven into the fabric of the text. Their use enhances the geometric flavor of the text. In Chapter 1, for instance, linear transformations provide a dynamic and graphical view of matrix–vector multiplication.

### **Eigenvalues and Dynamical Systems**

Eigenvalues appear fairly early in the text, in Chapters 5 and 7. Because this material is spread over several weeks, students have more time than usual to absorb and review these critical concepts. Eigenvalues are motivated by and applied to discrete and continuous dynamical systems, which appear in Sections 1.10, 4.8, 4.9, and in five sections of Chapter 5. Some courses reach Chapter 5 after about five weeks by covering Sections 2.8 and 2.9 instead of Chapter 4. These two optional sections present all the vector space concepts from Chapter 4 needed for Chapter 5.

### **Orthogonality and Least-Squares Problems**

These topics receive a more comprehensive treatment than is commonly found in beginning texts. The Linear Algebra Curriculum Study Group has emphasized the need for a substantial unit on orthogonality and least-squares problems, because orthogonality plays such an important role in computer calculations and numerical linear algebra and because inconsistent linear systems arise so often in practical work.

## **PEDAGOGICAL FEATURES**

### **Applications**

A broad selection of applications illustrates the power of linear algebra to explain fundamental principles and simplify calculations in engineering, computer science, mathematics, physics, biology, economics, and statistics. Some applications appear in separate sections; others are treated in examples and exercises. In addition, each chapter opens with an introductory vignette that sets the stage for some application of linear algebra and provides a motivation for developing the mathematics that follows. Later, the text returns to that application in a section near the end of the chapter.

### **A Strong Geometric Emphasis**

Every major concept in the course is given a geometric interpretation, because many students learn better when they can visualize an idea. There are substantially more drawings here than usual, and some of the figures have never appeared before in a linear algebra text.

### **Examples**

This text devotes a larger proportion of its expository material to examples than do most linear algebra texts. There are more examples than an instructor would ordinarily present in class. But because the examples are written carefully, with lots of detail, students can read them on their own.

### Theorems and Proofs

Important results are stated as theorems. Other useful facts are displayed in tinted boxes, for easy reference. Most of the theorems have formal proofs, written with the beginning student in mind. In a few cases, the essential calculations of a proof are exhibited in a carefully chosen example. Some routine verifications are saved for exercises, when they will benefit students.

### Practice Problems

A few carefully selected Practice Problems appear just before each exercise set. Complete solutions follow the exercise set. These problems either focus on potential trouble spots in the exercise set or provide a “warm-up” to the exercises, and the solutions often contain helpful hints or warnings about the homework.

### Exercises

The abundant supply of exercises ranges from routine computations to conceptual questions that require more thought. A good number of innovative questions pinpoint conceptual difficulties that I have found on student papers over the years. Each exercise set is carefully arranged, in the same general order as the text; homework assignments are readily available when only part of a section is discussed. A notable feature of the exercises is their numerical simplicity. Problems “unfold” quickly, so students spend little time on numerical calculations. The exercises concentrate on teaching understanding rather than mechanical calculations.

### True/False Questions

To encourage students to read all of the text and to think critically, I have developed 300 simple true/false questions that appear in 33 sections of the text, just after the computational problems. They can be answered directly from the text, and they prepare students for the conceptual problems that follow. Students appreciate these questions—after they get used to the importance of reading the text carefully. Based on class testing and discussions with students, I decided not to put the answers in the text. (The *Study Guide* tells the students where to find the answers to the odd-numbered questions.) An additional 150 true/false questions (mostly at the ends of chapters) test understanding of the material. The text does provide simple T/F answers to most of these questions, but it omits the justifications for the answers (which usually require some thought).

### Writing Exercises

An ability to write coherent mathematical statements in English is essential for all students of linear algebra, not just those who may go to graduate school in mathematics. The text includes many exercises for which a written justification is part of the answer. Conceptual exercises that require a short proof usually contain hints that help a student get started. For all odd-numbered writing exercises, either a solution is included at the back of the text or a hint is given and the solution is in the *Study Guide*, described below.