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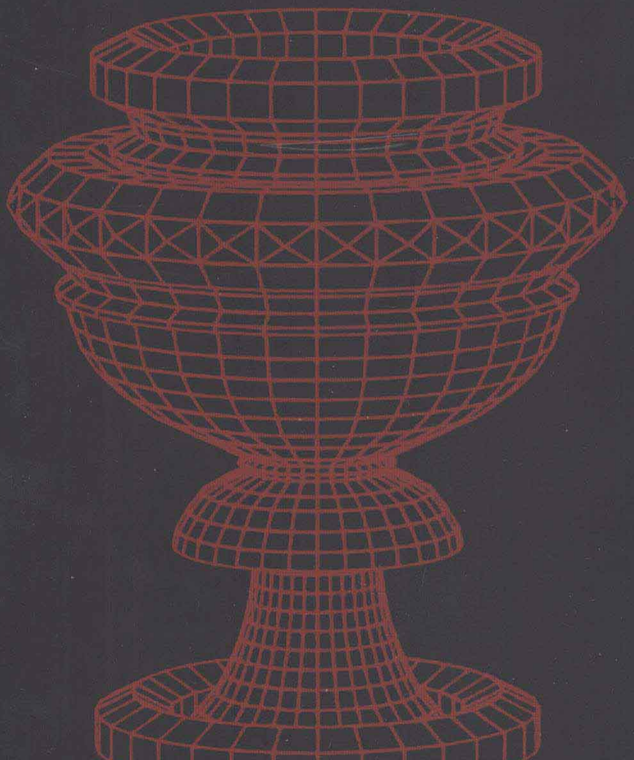
SURFACES IN CAGD '84



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SURFACES IN CAGD '84

Proceedings of a conference held at the
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F.R.G., 12–16 November 1984,
organized by R.E. Barnhill, W. Boehm and J. Hoschek

Edited by

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SURFACES IN CAGD '84

Computer Aided Geometric Design 2 (1985) v
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Preface

"Surfaces in Computer Aided Geometric Design" is an emerging field with interesting research problems and important applications. This subject draws on many different topics from Mathematics through Computer Science and Engineering. During the week of 12–16 November 1984 the Second International Symposium on "Surfaces in CAGD" was held at the Mathematics Institute, Oberwolfach, F.R. Germany. Some 40 speakers and 60 participants were selected from a large group of interested people to emphasize both breadth and depth of topics. Thus the participants came from universities, government and industrial laboratories and represented both theoretical and practical views of the subject. The chairmen of the meeting were R.E. Barnhill (Salt Lake City), W. Boehm (Braunschweig), and J. Hoschek (Darmstadt).

The spirit of the Oberwolfach meetings, to discuss current, ongoing research, is reflected in this Volume. Several of the papers herein are "Progress Reports" reflecting the current state of ongoing topics. This informality led to stimulating discussions.

The term "Computer Aided Geometric Design" was invented by R.E. Barnhill and R.F. Riesenfeld in 1974 to describe the more mathematical aspects of Computer Aided Design, an engineering discipline dedicated to the automation of design processes. The term "Surfaces in Computer Aided Geometric Design" was used by R.E. Barnhill, W. Boehm, and G. Farin in 1981 to describe the emerging emphasis on the more complex topic of surfaces, as distinct from curves. (Curves are still useful, too, of course.) The First International Symposium on Surfaces in CAGD, chaired by W. Boehm and J. Hoschek, was held at Oberwolfach in April, 1982 and the Proceedings were edited by R.E. Barnhill and W. Boehm and published by North-Holland.

The present Volume can be described in terms of research topics as follows: a survey by Barnhill is followed by five papers on approximations defined over triangles by Farin, Sablonniere, Chang et al., Gregory, and Sederberg. (Most of these papers use Bernstein–Bézier representations.) The next topic is bivariate B-splines with papers by Boehm, Prautzsch, and Dahmen et al. Two papers on general bivariate surfaces by Franke and Hoschek are followed by a paper on rectangular Coons patches by Worsey, which is followed by six papers on tensor product surfaces by Fritsch et al., Nowacki et al., Lyche et al., Boehm, Lasser, and Piegl. Properties and applications of surfaces are discussed in the eight papers by Brunet, Pratt, Dokken, Strasser et al., Houghton et al., Stead et al., Rabien, and Grieger. Two papers on curves by Hagen and Schumaker et al. conclude the Volume.

We would like to thank the participants for their many useful contributions to the success of the meeting and the subsequent papers. We also thank the Referees, who have improved many of the papers which appear here. Finally, we are grateful to Prof. Dr. Martin Barner, Director of the Mathematics Research Institute, for the opportunity of holding our research symposium at Oberwolfach.

We expect to hold the next conference on Surfaces in CAGD in February 8–14, 1987 at Oberwolfach and we look forward with anticipation to discussing new advances in this rapidly developing subject.

Robert E. BARNHILL
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Contributions

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Surfaces in computer aided geometric design: A survey with new results

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Presented at Oberwolfach 12 November 1984

Abstract. 'Surfaces in Computer Aided Geometric Design' focuses on the representation and design of surfaces in a computer graphics environment. This new area has the dual attractions of interesting research problems and important applications. The subject can be approached from two points of view: The design of surfaces which includes the interactive modification of geometric information and the representation of surfaces for which the geometric information is relatively fixed. Design takes place in 3-space whereas representation can be higher dimensional. 'Surfaces in CAGD' can be traced from its inception in rectangular Coons patches and Bezier patches to triangular patches which are current research topics. Triangular patches can interpolate and approximate to arbitrarily located data and require the preprocessing steps of triangulation and derivative estimation. New contouring methods have been found using these triangular patches. Finally, multidimensional interpolation schemes have been based on tetrahedral interpolants and are illustrated by surfaces in 4-space by means of color computer graphics.

Keywords. Surfaces, interpolation, approximation, design of surfaces, representation of surfaces, Coons patches, Bézier patches, triangular patches, contouring, multidimensional surfaces, computer graphics.

Contents

Preface

This article is a survey of an emerging subject, 'Surfaces in Computer Aided Geometric Design'. The purpose of this article is to present some of the fundamental concepts of our subject and to provide pointers to other work, enabling the reader to pursue the subject further. (In our subject many of the results are very new and others are 'folklore', making scholarly study difficult.)

Introduction and history

Significance of surfaces in CAGD

Most scientific representation of information requires approximations at some level. The approximation might occur at the level of equations that model the physical reality, or at the level of the numerical solution of these equations.

As in any science, for creating surfaces one has some quantitative data (such as scientific measurements) and some qualititative information (such as intuition of a 'good' shape). The quantitative data can be thought of as 'hard' data such as given positions and tangents. The qualitative data may be thought of as 'soft' information such as the desired shape. The philosophy for the construction of surfaces can be either *interpolation* or *approximation*. Interpolation means that one matches the given data exactly and approximation, a more general term, means one nearly matches the data. This dichotomy is discussed at some length in P.J. Davis' book [Davis '75].

Interpolation and approximation

At the most general level the tools employed to create surfaces include differential geometry, numerical analysis and computer graphics. Differential geometry is used to define surfaces. The spirit of numerical analysis is used to define surface interpolation methods to display surface forms efficiently by means of computer graphics [Barnhill '83b]. Computer graphics itself is an important research area which has undergone much growth in the past few years [Newman, Sproull '79; Foley, Van Dam '82]. Computer graphics illustrations play a central role in understanding and evaluating surfaces. A graphical capability that is tailored to surface schemes makes possible an immediate presentation of results with minimal interaction by the user. This wedding of mathematics and technology makes the subject more useful and more difficult.

History: Surfaces in Computer Aided Geometric Design

The representation and approximation of surfaces in a computer graphics environment may be considered to have been launched by two pioneers: S.A. Coons and P. Bézier. Coons' surfaces [Coons '64] and Bézier's [1] surfaces [Bézier '66, '67] each consist of a network of 'patches' which have a rectilinear topology. Coons' patches match exactly certain information (namely, whole curves of data). Bézier's surface methods have the different flavor that some data are matched exactly and the rest are approximated. Thus Coons' patches are a form of interpolation and Bézier's patches are a form of approximation which corresponds at a high level to Davis' dichotomy of interpolation and approximation. Specifically, Coons' 'blending functions' are the basis functions for Hermite interpolation and Bézier's blending functions are the basis functions for Bernstein approximation. [Barnhill '82] briefly surveys Coons patches and [Barnhill '85] and [Farin '85b] are preparing extensive surveys of Coons and Bezier patches, respectively.

Both Coons and Bézier were working in engineering environments when they discovered their patch methods. In order for mathematicians to analyze their methods, the underlying structures of the methods had to be recognized. As we shall see, their basic methods have been generalized and improved in various ways.

[1] Bézier and de Casteljau [de Casteljau '59, '63] independently developed equivalent curve and surface schemes now known only under Bézier's name.

W.J. Gordon [Gordon '69a] discovered that Coons patches have the powerful underlying algebraic structure of forming distributive lattices. Gordon described Coons' patches as Boolean sums of lower-dimensional 'projectors' which were themselves interpolants to lower-dimensional information.

At about the same time 'Gordon surfaces' [Gordon '69b, '69c] consisting of a network of patches were created. Gordon surfaces blend together a given rectilinear network of curves. The blending can be achieved, for example, with univariate Lagrange interpolants or interpolatory splines.

A generalization of Coons' original patches was also necessary for those situations in which four-sided topology cannot be assumed. [Barnhill, Birkhoff, Gordon '73] initiated triangular Coons' patches for the case of *arbitrarily located* information. This innovation created many new surface possibilities. These triangular methods have a more complicated data structure through which they solve the more complex problem of interpolation to more general data. Subsequently additional triangular patches have been discovered [Barnhill '83a, '83b; Nielson, Franke '83].

There has been a parallel set of developments for Bézier's methods:

(1) Gordon showed how Coons' patches could be analyzed mathematically. The corresponding discovery for Bézier's patches was done by Forrest [Forrest '72] who showed that Bézier curves and surfaces could be considered as Bernstein polynomial approximations. This recognition has made possible the discovery of many important features of Bézier approximations, such as the convex hull property and the variation diminishing property.

(2) The analogue to Gordon surfaces is a network of rectangular Bézier patches: tensor product B-splines were discussed in [Gordon, Riesenfeld '74].

(3) The analogue to the Barnhill, Birkhoff, and Gordon triangular Coons patch is the triangular Bézier patch which, for an arbitrary triangle, was discovered by Farin [Farin '80][2]. Farin's generalization has opened up many possibilities for the creation of new triangular interpolants as well as useful descriptions of known triangular interpolants. In fact, the Bézier method has become the starting point for generalizations that develop piecewise polynomial schemes with desired geometric properties, as is mentioned by several authors in this Volume.

The history of Coons patches and Bézier patches is summarized in Fig. 1. (The idea of this figure was conceived jointly with G. Farin who also made the drawing).

Computer Aided Geometric Design

The term 'Computer Aided Geometric Design' was invented by R.E. Barnhill and R.F. Riesenfeld in 1974 to describe the mathematical aspects of Computer Aided Design (hence the word 'geometric'). The term first appeared as the title of the symposium held at The University of Utah and the subsequent book published by Academic Press. Computer Aided Geometric Design focuses on design. In order to recognize the need for a new emphasis on representation and to focus on surfaces instead of curves, the new term 'Surfaces in Computer Aided Geometric Design' was coined by Barnhill, Boehm and Farin in 1981.

Surfaces in Computer Aided Geometric Design

A number of additional significant changes, central to the direction of the field, are embodied in the new name, 'Surfaces in Computer Aided Geometric Design'. Let us make these explicit here.

[2] Bézier patches over equilateral triangulations are given in [Sabin '76].

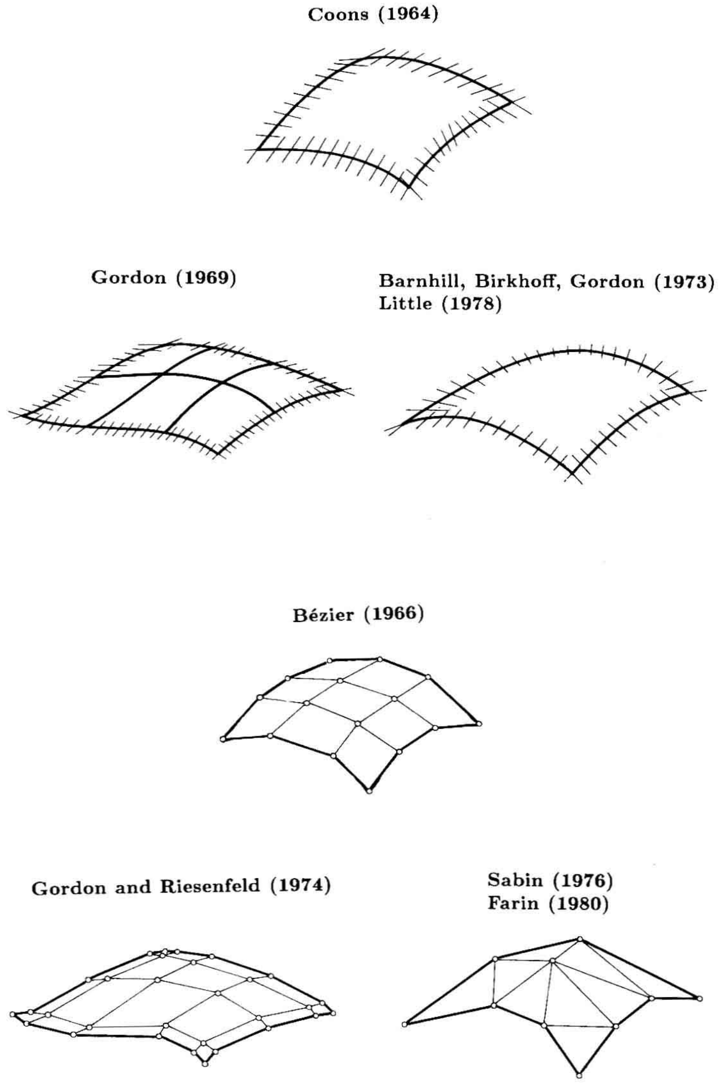

Fig. 1. History of Coons patches and Bézier patches.

(1) The research focuses on surfaces, not on curves.

(2) The surfaces, moreover, need not be built up from curves.

(3) Geometric data for surfaces can be arbitrarily located. (Surfaces used in practice have usually been based on rectangularly structured tensor product data.)

(4) Multidimensional surfaces are investigated.

Environment

Scholarly environment for a new subject

Disciplines that have strong technological components tend to be pursued in fragmentary ways with each problem treated on an ad hoc basis. 'Surfaces in CAGD' is an example of such a discipline. The subject can be made more scientific and integrated by means of research, training of new professionals in the field [Keyworth '83], collaborations, research symposia such as this one, books, and journals. Several books have summarized the research in this area: *Computer Aided Geometric Design*, edited by Barnhill and Riesenfeld in 1974, *Surfaces in Computer Aided Geometric Design*, edited by Barnhill and Boehm in 1983 and the *Surfaces* issue of the *Rocky Mountain Journal of Mathematics*, edited by Barnhill and Nielson in 1984. The new journal, *Computer Aided Geometric Design*, is devoted to recent research in this area.

Surfaces

Choice of surface form: Applications

Surfaces in Computer Aided Geometric Design have many applications, including fitting experimental data, tables of numbers and discretized solutions of differential equations; the design of aircraft, cars, and many other objects; and modeling human organs and robots. The term 'Computer Aided Design/Computer Aided Manufacturing' (CAD/CAM) is used to describe some of these applications, particularly in engineering. The choice of the surface form depends upon the application, that is, there is no single solution for all problems. The variety of applications is so great that there cannot be a universal panacea. For example, the surface form used to model the human heart is unlikely to have the correct properties for modeling a car body. Consequently, we shall consider several families of methods in both interpolation and approximation senses. We use the term 'surface modeling' to describe all applications since in all cases the mathematics describes a physical model.

Design and representation of surfaces

'Surfaces in CAGD' has two main categories: the Design of Surfaces and the Representation of Surfaces. Design of Surfaces involves making interactive changes in surfaces and displaying the surface in real-time. Representation of Surfaces involves using information derived elsewhere and of viewing the surface in order to understand its properties. These two categories have some common and some different features.

The features common to both Design and Representation include:

(1) Some smoothness is desirable. (This smoothness might be C^0, C^1, or C^2 continuity or might be 'visual continuity' of some order.)

(2) Shape fidelity must be satisfied. (This may be somewhat vague, such as a designer's idea of 'sweetness' or a geophysicist's view of what a surface should look like.)

(3) Methods may be either local or global. (With local methods the evaluation of the surface depends only on nearby data.)

(4) Computer graphics are useful.

Features that differ for Design and Representation include:

(1) The data can be modified and, possibly augmented for Design. The data for Representation usually are fixed and are expensive to obtain, e.g., results from wind tunnel tests. (However, the representation used can affect the design of the experiments.)

(2) Design surfaces are usually in three-space ('three-dimensional surfaces'), but Representation can take place in n-space.

(3) Finally, Design involves computer graphics allowing real-time modification so that the designer can get immediate feedback, whereas Representation involves viewing surfaces in order to understand them but not necessarily to make interactive changes in them. These functions may be seen as editing surfaces and viewing surfaces, respectively.

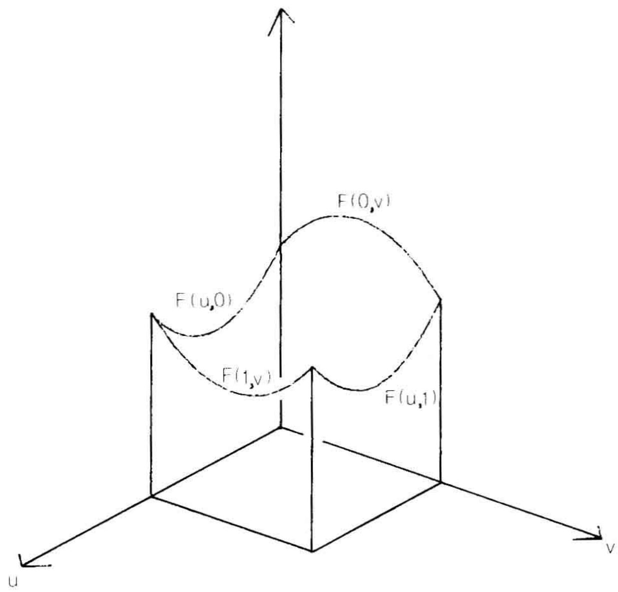

Fig. 2. Four data curves for the bilinearly blended Coons patch.

Coons patches and Bézier patches

Rectangular Coons patches

We now discuss Coons patches and Bezier patches. As mentioned above, patches can be either rectangular or triangular. Let us begin with rectangular Coons patches. The first case is the bilinearly blended Coons patch which interpolates to four curves as shown in Fig. 2. Gordon pointed out that this patch can be written as a Boolean sum of linearly ruled lofting interpolants, more precisely, if

$$P_1 = P_1 F = (1-u)\ F(0, v) + u\ F(1, v)$$

and P_2 is defined analogously, then the Boolean sum defined by

$$P_1 \oplus P_2 = P_1 + P_2 - P_1 P_2$$

interpolates to the four boundary curves. We observe the nice interplay between algebra and geometry here: one can build up the idea of the Boolean sum by looking at P_1, P_2, and P_1P_2, thus using geometry, and then verify the correctness of the solution by direct substitution, thus using algebra.

Next let us consider the bicubically blended Coons patch which can be built up algebraically in an analogous way, namely, by means of the Boolean sum of univariate projectors. The projector P_1 is given by

$$P_1 F = h_0(u)\ F(0, v) + h_1(u)\ F(1, v) + \bar{h}_0(u)\ F_{1,0}(0, v) + \bar{h}_1(u)\ F_{1,0}(1, v)$$

where the blending functions h_0, h_1, $\bar{h}_0$, and $\bar{h}_1$ are the cubic Hermite basis functions and $F_{1,0}$ means partial derivative with respect to u. The projector $P_2 F$ is defined similarly in the second variable v. The term $P_1 P_2 F$ involves the following data:

positions,	tangents,
tangents,	twists.

The twists, which are (1, 1)-derivatives, can cause problems in using this patch. Solutions to this problem are given in [Gregory '74] and in [Barnhill, Brown, Klucewicz '78]. An application of their research to reducing surface oscillations by varying twists is given by Brunet in this Volume.

We recently created the multidimensional compatibly corrected C^1 Coons patch for a higher-dimensional representation problem [Barnhill, Worsey '84]. The generalization to C^2 Coons patches is discussed by Worsey in this Volume.

Coons patches are 'transfinite'; this term, introduced by Gordon [Gordon '71] connotes that whole curves of data are interpolated. These data can be discretized leading to finite dimensional interpolants some of which are called 'serendipity elements' in the finite element literature. An example is the discretization of the bicubically blended Coons patch to the standard 16 degree of freedom bicubic patch obtained by replacing $F(u, 0)$, $F_{0,1}(u, 0)$, etc. by their respective cubic Hermite interpolants.

Triangular Coons patches

Triangular Coons patches were initiated by Barnhill, Birkhoff, and Gordon [Barnhill et al. '73] who considered the C^1 case with the corresponding cubic Hermite projectors along parallels to each side, that is, P_1 is the cubic Hermite lofting interpolant along parallels to side 1 etc. The 'BBG triangle' is a family of interpolants formed by taking Boolean sums of the three lofting interpolants P_1, P_2, and P_3. Twist incompatibilities inherent in all Boolean sums must again be resolved in order to produce suitable schemes. Little [Little '78] made the very important step of generalizing the 'BBG' schemes to an arbitrary triangle [Barnhill, Little '84], a key concept being a calculus for functions of barycentric coordinates.

Other transfinite triangular interpolants have subsequently been discovered, including Nielson's radial schemes [Nielson '79], Gregory's symmetric schemes (which are generalized to n-dimensional simplices in this Volume), and Brown, Dube, and Little's convex combinations. Recently Alfeld and Barnhill [Alfeld, Barnhill '84] constructed a C^2 BBG scheme. Finally, Little has devised a trivariate C^1 BBG scheme [Barnhill, Little '84].

Bézier patches

Bernstein–Bézier approximations have recently become very popular and no fewer than 1/4 of the titles at this symposium contain Bézier's name. Bernstein–Bézier patches interpolate some data and approximate others. This representation is described by a 'net' of 'control vertices', where the vertices of the net are the coefficients of the Bernstein basis functions. (See Fig. 1 in Farin's article in this Volume.)

Bézier patches (like Coons patches) can be either rectangular or triangular. Rectangular Bézier patches are tensor products, so their properties follow from the univariate case. (For additional information on tensor products, as well as on CAGD in general, see [Boehm, Farin, Kahmann '84].) Therefore, we shall content ourselves with a brief introduction to triangular Bézier patches.

A necessary tool for the construction of a triangular interpolant over an arbitrary triangle is the concept of barycentric coordinates. The barycentric coordinates of the arbitrary point P in a triangle with vertices 1, 2, 3 are given by $b_i = A_i/A$ where A is the area of the triangle and A_i is the area of the subtriangle opposite vertex i, $i = 1, 2, 3$. This geometry for barycentric coordinates appears in Fig. 3. The definition of barycentric coordinates implies that they are non-negative if P is in the triangle and that they sum to one.

The triangular Bernstein polynomial is given by

$$\sum_{i+j+k=n} \frac{n!}{i!j!k!} b_1^i b_2^j b_3^k V_{i,j,k}$$

where the $V_{i,j,k}$ are (vector-valued) 'control vertices'. Imposing continuity between Bézier