Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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Author

Chris Preston
Universität Bielefeld, USP-Mathematisierung
4800 Bielefeld, Federal Republic of Germany

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for Os and Tam

The elegant body of mathematical theory pertaining to linear systems (Fourier analysis, orthogonal functions, and so on), and its successful application to many fundamentally linear problems in the physical sciences, tends to dominate even moderately advanced University courses in mathematics and theoretical physics. The mathematical intuition so developed ill equips the student to confront the bizarre behaviour exhibited by the simplest of discrete nonlinear systems, such as the equation $x_{n+1} = ax_n(1-x_n)$ *. Yet such nonlinear systems are surely the rule, not the exception, outside the physical sciences.*

Bob May in *Simple mathematical models with very complicated dynamics* in *Nature*, Vol. 261, June 1976.

These are some notes on the iterates of maps on an interval, which we hope can be understood by anyone who has had a basic course in (one-dimensional) real analysis. The main reason for writing this account is as an attempt to make the very beautiful mathematics behind the *bizarre behaviour exhibited by the simplest of discrete nonlinear systems* accessible to as wide an audience as possible.

Parts of these notes have appeared as Volumes 34 and 37 in the series: *Materialien des Universitätsschwerpunktes Mathematisierung* from the Universität Bielefeld, and I would like to thank the USP Mathematisierung for their support. Thanks also to David Griffeath for some pertinent comments on the text and to Bob May and Alister Mees for getting me interested in this subject.

Bielefeld
October 1982

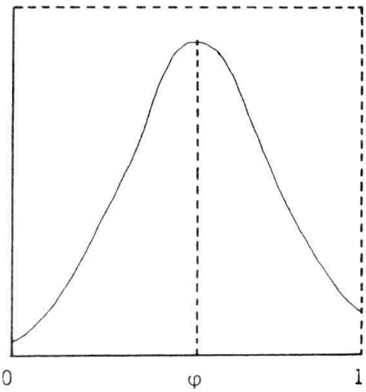
Chris Preston

ITERATES OF MAPS ON AN INTERVAL - CONTENTS

1. INTRODUCTION

Suppose we are studying some physical or biological system on which we make measurements at regular intervals (say once a year or every ten seconds). If we are just measuring a single quantity then the n th. measurement can be represented by a real number x_n . The data we thus obtain is a sequence $x_0, x_1, \ldots, x_m$ of real numbers, where of course $m+1$ is the number of measurements which are made. A very simple mathematical model of such a system is obtained by assuming that x_{n+1} is only a function of x_n , and that this function does not depend on n . That is, we assume there is a function $f : X \to X$ so that $x_{n+1} = f(x_n)$ for all $n \geq 0$, where $X \subset \mathbb{R}$ represents the set of possible values which can be registered by our measuring apparatus. Thus if a measurement at time 0 gave a value $x \in X$ then the model predicts that a measurement at time n would register a value of $f^n(x)$, where $f^n : X \to X$ is given inductively by $f^0(x) = x$, $f^1(x) = f(x)$ and $f^n(x) = f(f^{n-1}(x))$. With such a model we are therefore interested in the iterates $\{f^n\}_{n\geq 0}$ of the function f .

Typical functions which have been used (for example as models in population biology; see, for instance, May (1976)) are the functions $f_\mu : [0,1] \to [0,1]$ (with $0 < \mu \leq 4$) and $g_r : [0,1] \to [0,1]$ (with $r > 1$) given by $f_\mu(x) = \mu x(1-x)$ and $g_r(x) = rx \exp(1-rx)$. The important feature common to the functions in these two families is that they all look something like the picture at the top of the next page. More precisely, each of the functions has a unique maximum in $(0,1)$ and is strictly increasing (resp. strictly decreasing) to the left (resp. right) of this point. Thus the f_μ and g_r are elements of the set S



consisting of those continuous functions $f : [0,1] \to [0,1]$ for which there exists $\varphi \in (0,1)$ such that f is strictly increasing on $[0,\varphi]$ and is strictly decreasing on $[\varphi,1]$.

These notes will be concerned with the question: What kinds of behaviour can be exhibited by the iterates of a function in S ? With the help of a programmable calculator the reader can soon convince himself that for some functions in S the behaviour of the iterates is very simple, while for others it is extremely complicated. Consider, for example, the family of functions $\{f_\mu\}_{0<\mu\leq 4}$ in S given by $f_\mu(x) = \mu x(1-x)$. For each of the five parameter values $\mu = 2.5$, 3.1 , 3.569946 , 3.8291 and 4 compute the first 10000 or so terms of the orbit $\{f_\mu^n(x)\}_{n\geq 0}$ corresponding to a "randomly" chosen starting point $x \in [0,1]$ (where we have written and will continue to write f_μ^n instead of $(f_\mu)^n$). Repeat this several times with other "random" starting points. The following "facts" can then be observed:

<u>$\mu = 2.5$</u> . 0.6 is a fixed point of f_μ (i.e. $f_\mu(0.6) = 0.6$) and all

the points in $(0,1)$ are attracted to this point, i.e. $\lim_{n\to\infty} f_\mu^n(x) = 0.6$ for all $x \in (0,1)$. (Note however that 0 and 1 are not attracted to 0.6 since $f_\mu(0) = f_\mu(1) = 0$.)

$\underline{\mu = 3.1}$. 0.55801.. is a periodic point with period 2 and practically all the points in $[0,1]$ are attracted to the period orbit $\{0.55801.., 0.76456..\}$. (If $f : [0,1] \to [0,1]$ then $x \in [0,1]$ is called periodic if $f^m(x) = x$ for some $m \geq 1$; the smallest $m \geq 1$ with this property is called the period of x . If x is periodic with period m then we say $y \in [0,1]$ is attracted to the periodic orbit $\{x, f(x), \ldots, f^{m-1}(x)\}$ if $\lim_{n\to\infty} f^{mn}(y) = f^k(x)$ for some $0 \leq k < m$.)
In the present case "practically all" means except for the three points 0 , 1 and 0.6774.. . ($0.6774.. = \frac{2.1}{3.1}$ is the unique fixed point of f_μ in $(0,1)$.)

$\underline{\mu = 3.569946}$. The orbit of a randomly chosen $x \in [0,1]$ appears to be attracted to an orbit which is almost, but not quite, periodic. More precisely, the following can be observed: For $m \geq 1$ let $x_n^m \in [0,1]$ be obtained by rounding off $f_\mu^n(x)$ to m decimal places. Then for each $m \geq 1$ there is a periodic sequence $\{y_n^m\}_{n\geq 0}$ (i.e. $y_{n+k}^m = y_n^m$ for some $k \geq 1$ and for all $n \geq 0$) such that for each randomly chosen x there exists $j \geq 0$ with $x_{n+j}^m = y_n^m$ for all $n \geq 0$. However, the period of the sequence $\{y_n^m\}_{n\geq 0}$ grows very rapidly with m . (For $m = 1$ the period is 16 , for $m = 2$ it is 128 .) Another feature which can be seen with this value of μ is that the orbits of points near a randomly chosen x look very similar to the orbit of x . (For each

$\varepsilon > 0$ it is possible to find $\delta > 0$ so that if $|x-y| < \delta$ then $|f_\mu^n(x)-f_\mu^n(y)| < \varepsilon$ for all $n \geq 0$.)

$\underline{\mu = 3.8291}$. 0.15747.. is a periodic point with period 3 and almost all points in [0,1] are attracted to the periodic orbit {0.15747.., 0.50802.., 0.95702..} . We will later see that "almost all" can here be taken in the sense of Lebesgue measure on [0,1] , but that the situation is more complicated than in the first two cases in that there are uncountably many points which do not get attracted to this periodic orbit.

$\underline{\mu = 4}$. The orbit of a randomly chosen $x \in [0,1]$ appears to be completely "chaotic". Moreover, the orbit of x looks nothing like the orbit of a point chosen "randomly" in any small neighbourhood of x .

The above numerical examples suggest that there are at least three different kinds of behaviour which can be exhibited by the iterates of a function in S . These are:

(1) There is a periodic orbit which attracts "almost all" of the points in [0,1] . (This case occurs for $\mu = 2.5$, 3.1 and 3.8291 .)

(2) A typical orbit appears to be completely random; there is a sensitive dependence to initial conditions. ($\mu = 4$)

(3) There is a "strange attractor" which attracts "almost all" of the points in [0,1] ; there is no sensitive dependence to initial conditions. ($\mu = 3.569946$)

These three types of behaviour will provide the key to understanding the iterates $\{f^n\}_{n\geq 0}$ of a function $f \in S$. It turns out that for functions in an important subset of $\mathcal{S}$ (which includes the families $\{f_\mu\}_{0<\mu\leq 4}$ and $\{g_r\}_{r>1}$) the three cases (1), (2) and (3) completely classify the behaviour which can occur. For a general function in $\mathcal{S}$ the situation can be more complicated, but (1), (2) and (3) are still the basic prototypes for the possible kinds of behaviour.

The mathematical results which are behind the statements in the above paragraph are due to Guckenheimer and Misiurewicz (Guckenheimer (1979), Misiurewicz (1980)). It is the object of these notes to give an elementary account of these and other results on the iterates of functions in $\mathcal{S}$. We have tried to make this account understandable to anyone who knows the basic facts about continuous and differentiable functions of one real variable (as can be found, for example, in the first five chapters of Rudin: Principles of Mathematical Analysis (1964)). In fact one of the main reasons for writing these notes is as an attempt to make some very beautiful mathematics accessible to as wide an audience as possible.

Before giving an outline of what is contained in the various sections of these notes we will make a couple of general remarks.

1. A lot of the interest in the behaviour of maps on an interval was kindled by the review article of May in *Nature* (May (1976)). The reader is strongly recommended to look at this article.

2. A second strong recommendation is to study the book by Collet and Eckmann (1980) called *Iterated maps on the interval as dynamical systems*. As its title suggests, it is concerned with much the same material as we will consider and in particular it gives an account of the results of Guckenheimer and Misiurewicz. Our justification for writing the present set of notes is that many of our proofs are simpler than the corresponding

ones in Collet and Eckmann. In any case, a second account will have served some useful purpose if it increases the number of people who are interested in the iterates of maps on an interval.

3. We only consider the behaviour of the iterates of a single function. However, in practice it is often more important to study how this behaviour changes when we vary some parameter. For example, how does the behaviour of the iterates of f_μ change as μ increases in the interval (0,4] ? There has been a lot of interest in such questions in the last couple of years; this interest started with the discovery by Feigenbaum (Feigenbaum (1978), (1979)) that the successive bifurcations in any reasonable one-parameter family of functions from $\mathcal{S}$ exhibit a remarkable quantitative universality (in that the rates at which the bifurcations occur involve constants which are common to all such families). Unfortunately, the mathematics needed to handle these problems rigorously is way beyond what we intend to use here, and so we will not be able to study this topic. The reader is recommended to look at Hofstadter's column (*Metamagical Themas*) in the November 1981 *Scientific American*. Anyone who wants to see what kind of mathematics is involved in this area can also look at Collet, Eckmann and Lanford (1980).

4. We have made several statements involving "almost all" of the points in [0,1] . For most of these notes this will have a topological, rather than a measure-theoretical, meaning. In Sections 3 and 4 we will consider a set to contain "almost all" of the points in [0,1] if it contains a dense open subset of [0,1] , and in the following sections if it contains a countable intersection of dense open subsets (i.e. if it is a residual subset of [0,1] in the terminology of the Baire category theorem). The main reason for taking this approach is that it greatly simplifies a lot of the proofs; it also allows a lot of the notes to be read by someone

who has had no measure theory. In Section 9 we consider a measure-theoretic version of the main result obtained in the previous sections, and then "almost all" will mean in the sense of Lebesgue measure.

5. The fact that the functions in $\mathcal{S}$ are defined on the interval $[0,1]$ is not important. Suppose $f : [a,b] \longrightarrow [a,b]$ is a continuous function for which $\xi \in (a,b)$ exists so that f is strictly increasing on $[a,\xi]$ and is strictly decreasing on $[\xi,b]$. Then we can define $g \in \mathcal{S}$ by $g(x) = (b-a)^{-1}\{f((1-x)a+xb)-a\}$ (i.e. by making a linear change of variables), and any properties we are interested in will be invariant under this transformation.

6. Our description of what happens to the iterates $\{f_\mu^n\}_{n\geqq 0}$ in the case when $\mu = 3.569946$ was not completely honest. This value of μ is only an approximation to the value of the parameter which really gives the behaviour we described. (The "correct" value of μ lies between 3.5699456 and 3.5699457 .) In fact, when $\mu = 3.569946$ then there is a periodic orbit with period 23×2^{10} which attracts "almost all" of the points in $[0,1]$. (In Section 10 we will explain how the parameter value $\mu = 3.569946$ was chosen.)

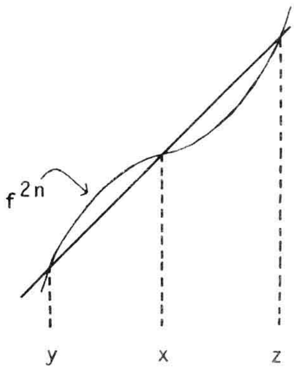
7. There are many topics involving the iterates of a single function from $\mathcal{S}$ which we do not consider in these notes. Perhaps the most important concerns the existence of absolutely continuous invariant probability measures. Let $f \in \mathcal{S}$ and μ be a probability measure on $([0,1],\mathcal{B})$, where $\mathcal{B}$ denotes the Borel subsets of $[0,1]$; μ is called invariant under f if $\mu(f^{-1}(F)) = \mu(F)$ for all $F \in \mathcal{B}$. A question which has received a lot of attention recently is: Which functions in $\mathcal{S}$ have an absolutely continuous (with respect to Lebesgue measure) invariant probability measure? (μ is absolutely continuous with respect

to Lebesgue measure if $\mu(F) = 0$ whenever the Lebesgue measure of F is zero.) The main result in this direction is due to Misiurewicz (Misiurewicz (1980)); an account of this result can be found in Collet and Eckmann (1980). Another topic which we do not consider is the problem of determining whether or not two given functions from $\mathcal{S}$ are topologically conjugate. Continuous functions $f, g : [0,1] \longrightarrow [0,1]$ are said to be topologically conjugate if there exists a homeomorphism $h : [0,1] \longrightarrow [0,1]$ such that $f = h^{-1} \circ g \circ h$. If $f = h^{-1} \circ g \circ h$ then we also have $f^n = h^{-1} \circ g^n \circ h$ for each $n \geqq 1$; thus if $f, g \in \mathcal{S}$ are topologically conjugate then the iterates of f and g will exhibit the same kind of topological behaviour. A well-known example of this is provided by the functions $f(x) = 4x(1-x)$ and the piecewise linear function $g(x) = \begin{cases} 2x & \text{if } 0 \leqq x \leqq \frac{1}{2} , \\ 2-2x & \text{if } \frac{1}{2} \leqq x \leqq 1 . \end{cases}$ (For these functions we have $f = h^{-1} \circ g \circ h$ with $h(x) = \frac{1}{2} + \frac{1}{\pi} \sin^{-1}(2\sqrt{x} - 1)$.) For an account of this subject the reader is again referred to Collet and Eckmann (1980).

We now outline what is contained in the various sections; at the end of each section there are some bibliographic notes to be found. The main results concerning the iterates of functions in $\mathcal{S}$ (Theorems 5.1 and 5.2) are stated in Section 5..It is possible for the reader to start at Section 5, and in order to make this easier we give an index of symbols at the end of the notes.

Section 2: Piecewise monotone functions Let $a, b \in \mathbb{R}$ with $a < b$ and let $f : [a,b] \longrightarrow [a,b]$ be continuous; f is called piecewise monotone if there exist $N \geqq 0$ and $a = d_0 < d_1 < \cdots < d_N < d_{N+1} = b$ such that f is strictly monotone on each of the intervals $[d_k, d_{k+1}]$, $k = 0,\ldots,N$. Section 2 deals with some elementary properties of such functions. Our interest in this class of functions lies in the fact that if $f \in \mathcal{S}$ then f^n is piecewise monotone for each $n \geq 1$. Moreover, it is often just as easy to obtain results for piecewise monotone functions as it is for functions in $\mathcal{S}$.

The section is mainly concerned with periodic points, and besides the usual classification of periodic points as being either stable, one-sided stable or unstable we also introduce the notion of a periodic point being "trapped": If x is periodic with period n then we say that x is trapped if there exist $y < x < z$ and $\delta > 0$ such that f^{2n} is increasing on the interval $[y-\delta, z+\delta]$ and $f^{2n}(y) \leq y$, $f^{2n}(z) \geq z$. The most important example of such a point is when a stable periodic point x is "trapped" between two unstable periodic points y and z:



We will see that if a periodic point x is trapped then so are all the points in the periodic orbit $[x] = \{x, f(x), \ldots, f^{n-1}(x)\}$, and thus it also makes sense to talk about a periodic orbit being trapped.

The main result of the section implies that if $f \in \mathcal{S}$ then f has at most one periodic orbit $[x] = \{x, f(x), \ldots, f^{n-1}(x)\}$ such that *(i)* $[x]$ is either stable or one-sided stable, *(ii)* $[x]$ is not trapped, and *(iii)* x is not a fixed point of f in $[0, \varphi)$ (where φ is the turning point of f). Moreover, if this orbit $[x]$ exists then for some $\delta > 0$ all the points in $(\varphi-\delta,\varphi)\cup(\varphi,\varphi+\delta)$ are attracted to $[x]$. This result is based on the proof of a theorem in Singer (1978). The existence or non-existence of the orbit $[x]$ will be important in the later sections for determining what kind of behaviour occurs for the iterates of f .

Section 3: Well-behaved piecewise monotone functions For a piecewise monotone function $f : [a,b] \longrightarrow [a,b]$ we let $A(f)$ denote the set of points in $[a,b]$ which are attracted to some periodic orbit of f ; it is easily seen that in fact

$$A(f) = \{ x \in [a,b] : \lim_{m\to\infty} f^{nm}(x) \text{ exists for some } n \geq 1 \} ,$$

and thus in some sense $A(f)$ consists of those points y in $[a,b]$ for which the orbit $\{f^n(y)\}_{n\geq 0}$ has a particularly simple behaviour. The aim of Section 3 is to find sufficient conditions under which $A(f)$ contains a dense open subset of $[a,b]$, i.e. under which a "typical" point in $[a,b]$ gets attracted to a periodic orbit. (See Remark 4 above.) We are also interested in knowing when $A'(f)$ contains a dense open subset of $[a,b]$, where $A'(f)$ is the set of points in $[a,b]$ which are attracted to either a stable or a one-sided stable periodic orbit.

In order to state what the main result of Section 3 says for functions in $\mathcal{S}$ let us fix $f \in \mathcal{S}$ with turning point φ and put

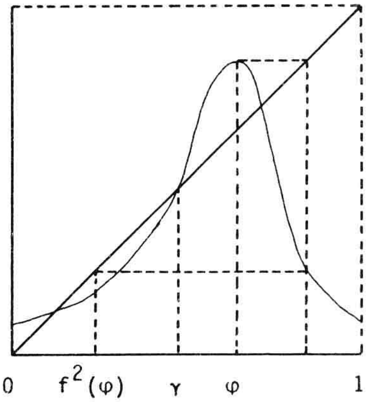
$$\gamma = \begin{cases} \text{the largest fixed point of } f \text{ in } [0,\varphi] & \text{if } f \text{ has a fixed point in } [0,\varphi] , \\ 0 & \text{otherwise.} \end{cases}$$

It is convenient to divide things into three cases:

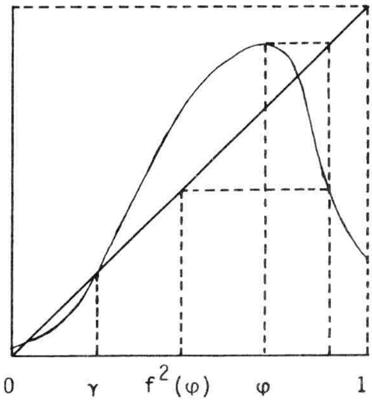
I. $f(\varphi) \leqq \varphi$,

II. $f(\varphi) > \varphi$ and $f^2(\varphi) < \gamma$,

III. $f(\varphi) > \varphi$ and $f^2(\varphi) \geqq \gamma$.



Case II



Case III

Case I is trivial and it is a simple matter to check that $A(f) = [0,1]$. Case II: Here we will see that $A(f)$ contains a dense open subset of $[0,1]$ provided

(1.1) f has a continuous second derivative in $(0,\varphi)\cup(\varphi,1)$ and $f'(x) \neq 0$ for all $x \in (0,\varphi)\cup(\varphi,1)$.

Case III: This is the most interesting situation. We will show that $A(f)$ contains a dense open subset of $[0,1]$ provided (1.1) holds and one of the following three conditions is satisfied:

(1.2) φ is attracted to a stable periodic orbit $[y]$;

(1.3) φ is attracted to a one-sided stable periodic orbit $[y]$ but $f^k(\varphi) \neq y$ for all $k \geqq 0$;

(1.4) the periodic orbit $[x]$ described in the main result of Section 2