Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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Geometry and Topology

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III Latin American School of Mathematics

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Preface

The III Latin American School of Mathematics (III ELAM) aimed to stimulate the development of the areas of geometry and topology in Latin America and to expand the interchange of ideas and contacts among mathematicians of this region initiated with the support of the OAS in the I and II Schools, held respectively in Brazil and Mexico. The III ELAM congregated more than 250 mathematicians (researches and students) most of them from Latin America. The members of the Organizing Committee were: C. Camacho, P. Schweitzer, M. do Carmo and J. Palis (Coordinator).

These Proceedings reflect one of the two main activities of the meeting, namely a series of research talks covering topics in Dynamical Systems, Differential Geometry, Foliations, Singularities of Mappings and Algebraic Topology. The other activity was a series of introductory courses in these subjects given by M. Peixoto, M. do Carmo, P.Schweitzer, J. Sotomayor and J. Adem, whose lecture notes are being or have been informally published by the Organizing Committee.

We hope that these Proceedings can contribute to the further progress of these areas of Mathematics within and outside Latin America.

To many of our colleagues, to several universities and research institute of Latin America and to the director of the host institution L. Dias we express our gratitude for their most valuable help.

We acknowledge the financial support of the Brazilian agencies CNPq, FINEP, FAPESP and CAPES, as well as the OAS.

Jacob Palis and Manfredo do Carmo

Rio de Janeiro, April, 1977

CONTENTS

EVALUATION OF SOME MAUNDER COHOMOLOGY OPERATIONS

By

José Adem and Kee Yuen Lam

Introduction

In this paper we present an explicit evaluation of primary, secondary and tertiary Maunder cohomology operations on complex projective spaces. These operations are related to divisibility properties of the Chern character, as was first discovered by Adams for primary operations ([2]) and later extended by Maunder to higher order operations ([14]).

If η is the canonical complex line-bundle over the N-dimensional complex projective space CP^N, we have

$$ch(\eta-1)^n = (e^{\omega}-1)^n = \omega^n + \Sigma_{t=1}^{\infty}\{n+t,n\}\omega^{n+t},$$

where the rational numbers $\{n+t,n\}$ are the Taylor coefficients and $\omega \in H^2(CP^N;Z)$ is the integral cohomology generator. According with Adams (loc. cit.) the number

$$(*) \qquad m(t)\{n+t,n\}$$

is an integer, where the numerical function m(t) is defined by

$$m(t) = \prod_p p^{[t/(p-1)]}$$

where p runs over the primer numbers.

Roughly speaking, to compute the higher order Maunder operations (mod 2) on the class ω^n is equivalent to determine the highest power of two contained in the integers (*). In section 3 we give explicit expressions that allow us to compute the 2-integers $2^t\{n+t,n\}$ mod 8 and this is enough for our purpose.

We obtain simple formulas in terms of binomial coefficients to evaluate the Maunder secondary and tertiary operations on the class ω^n (see (4.6), (4.17)). Their duals, denoted by $\Phi^{(2)}_{2r}$ for secondary and by $\Phi^{(3)}_{2r}$ for tertiary operations, are also evaluated (see (6.5), (7.12)). The secondary operations $\Phi^{(2)}_{2r}$ have been used to solve several problems. Through "accompany relations" they can be combined to yield non-trivial operations on real projective spaces, that are applied to solve non-immersion problems ([5],[6]). We do not know how to apply this procedure to the $\Phi^{(3)}_{2r}$. In general, examples of non-trivial tertiary operations on real projective spaces promise to be interesting.

We do not give any application of our results, with exception of this one, presented right here. Let $q = 2^{n+2}$ and $r = 2^n$, with $n \geq 2$. Our evaluation of tertiary operations in (7.12), gives

$$\Phi^{(3)}_{2r}\omega^q = \omega^{q+r} \qquad \text{(mod 2)},$$

in $H^*(CP^{q+r}/CP^{q-1};Z_2)$. Now, a result of Gitler and Milgram ([11]), shows that the operation $\Phi^{(3)}_{2r}$ on an integral class of dimension $\ell \leq 2r-7$ is a primary operation. This operation in our case becomes zero and we have proved that the stunted projective space CP^{q+r}/CP^{q-1} cannot be desuspended $2q-2r+7$ times.

Finally, we want to indicate that Dennis Phee Hurley has obtained in [16], evaluation results for a full family of modified Maunder dual operations of all orders. His work agrees with our theorem (7.12) and can be regarded as a generalization of the results given in [10], for secondary operations. However, it seems that he cannot obtain our theorem (4.17).

1. The Maunder operations

As defined in [14;p.753], let $C(3,r)$ be the chain complex $(3 \leq r)$

(1.1) $$C_3 \xrightarrow{d_3} C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0,$$

where each C_i is a free-graded left module over the mod 2 Steenrod algebra A, on generators described as follows:

C_0 with A-basis $\{c_0\}$,

C_1 with A-basis $\{c_1, c_{10}, c_{11}\}$,

C_2 with A-basis $\{c_2, c_{20}, c_{21}, c_{22}\}$,

C_3 with A-basis $\{c_3\}$,

where $\dim c_0 = 0$, $\dim c_i = 2r+i-1$ for $i = 1,2,3$, and $\dim c_{ni} = 3n-2i$, for $n = 1,2$. On these generators, the maps d_i are given by

(1.2) $$\begin{cases} d_1c_1 = (\chi Sq^{2r})c_0, \\ d_1c_{10} = Sq^{01}c_0, \\ d_1c_{11} = Sq^1c_0. \end{cases}$$

(1.3) $$\begin{cases} d_2c_2 = Sq^1c_1 + (\chi Sq^{2r})c_{11} + (\chi Sq^{2r-2})c_{10}, \\ d_2c_{20} = Sq^{01}c_{10}, \\ d_2c_{21} = Sq^1c_{10} + Sq^{01}c_{11} \\ d_2c_{22} = Sq^1c_{11}, \end{cases}$$

(1.4) $$d_3c_3 = Sq^1c_2 + (\chi Sq^{2r})c_{22} + (\chi Sq^{2r-2})c_{21} + (\chi Sq^{2r-4})c_{20}$$

where $Sq^{01} = Sq^2Sq^1 + Sq^1Sq^2$ and $\chi : A \longrightarrow A$ is the canonical antiautomorphism. Each map d_i is of degree zero and $d_2d_3 = d_1d_2 = 0$ follows from the relations

$$(1.5) \qquad Sq^1Sq^1 = 0,$$

$$(1.6) \qquad Sq^{01}Sq^{01} = 0,$$

$$(1.7) \qquad Sq^1Sq^{01} + Sq^{01}Sq^1 = 0,$$

$$(1.8) \qquad Sq^1(\chi Sq^{2r}) + (\chi Sq^{2r-2})Sq^{01} + (\chi Sq^{2r})Sq^1 = 0.$$

Associated with C(3,r) we have a pyramid of operations $\Phi^{s,t}$, for $3 \geq s > t \geq 0$, where each $\Phi^{s,t}$ is of $(s-t)^{\underline{th}}$ order. We recall that this set of operations verify the Maunder's axioms and that they satisfy the relations with the Chern character as stated in [14;Th.2].

Let $w \in H^2(CP^N;Z_2)$ be the mod 2 cohomology generator of the N-dimensional complex projective space, where N is a large enough number, and let w^n be the n-fold product of w. Sometimes it is necessary to work in a (2n−1)-connected complex (see [14;Th.1]). In this case, through the collapsing map, w^n is regarded as an element of $H^*(CP^N/CP^{n-1};Z_2)$. Set

$$(1.9) \qquad \varepsilon : C_0 \longrightarrow H^*(CP^N/CP^{n-1};Z_2),$$

as the A-map defined by $\varepsilon(c_0) = w^n$. For ε (or equivalently for w^n), we shall make explicit the primary, secondary and tertiary operations associated with the chain complex (1.1).

In order to motivate the computation developed in section 3, we advance the following. Let η be the canonical complex line bundle over CP^N, and $\mu = \eta - 1$. The n-fold product μ^n can be regarded as an element of the Grothendieck ring $\tilde{K}_c(CP^N/CP^{n-1})$ (see [3;p.622]). If $\omega \in H^2(CP^N;Z)$ represents the integral cohomology generator then, for the Chern character, we have $ch\mu = e^{\omega}-1$ and

(1.10) $$ch\mu^n = (e^{\omega}-1)^n.$$

2. A binomial identity

In this section we state the Jensen identity concerning binomial coefficients $\binom{p}{q}$. Let α,β,γ and n be positive integers. Then we have

(2.1) $$\sum_{k=0}^{n} \binom{\alpha+\beta k}{k}\binom{\gamma+\beta(n-k)}{n-k} = \sum_{i=0}^{n} \beta^i \binom{\alpha+\gamma+n\beta-i}{n-i}.$$

For a proof see [9].

3. Computation of $2^t\{n+t,n\}$ mod 8

Write $(e^{\omega}-1)^n = \sum_{t=0}^{\infty} \{n+t,n\}\omega^{n+t}$ where $\{n+t,n\}$ are the Taylor coefficients. The value of $2^t\{n+t,n\}$ mod 2^r must be computed in order to evaluate the Maunder operations on CP^{∞}. In this section we compute them mod 8. Computation mod 16 is also available. In general the task becomes more complicated modulo higher powers of 2.

All computations and divisibility arguments take place in $Z_{(2)}$, the ring of rational numbers with odd denominators. If $a,b \in Z_{(2)}$, $a \equiv b \pmod{2^r}$ means that $a-b = 2^r c$ for some $c \in Z_{(2)}$. This is the only kind of congruence considered in our paper.

First we assert that $2^t\{n+t,n\}$ is in $Z_{(2)}$ for all n and t. When $n = 1$, $2^t\{1+t,1\} = 2^t/(t+1)!$ is indeed in $Z_{(2)}$, because of the well known fact:

(3.1) The highest power of 2 dividing m! is $2^{m-\alpha(m)}$, where $\alpha(m)$ is the number of 1's in the dyadic expansion of m.

The general assertion that $2^t\{n+t,n\} \in Z_{(2)}$ now follows inductively from

Proposition (3.2). For all $n,t > 0$ we have

$$2^t\{n+1+t,\ n+1\} = \sum_{i=0}^{t} 2^i\{n+i,n\}2^{t-i}\{1+t-i,1\}. \tag{3.3}$$

The proof is by equating coefficients of ω^{n+1+t} in $(e^\omega-1)^{n+1} = (e^\omega-1)^n(e^\omega-1)$.

To compute $2^t\{n+t,n\}$ mod 8, we first do so for $n = 1$ in theorem (3.4) below. We then attack the general case inductively on n, using arguments based on (3.2) and the binomial identity (2.1).

Theorem (3.4). For every $t > 0$ the value of $2^t/(t+1)!$ mod 8 is given by

$$\frac{2^t}{(t+1)!} \equiv \binom{1+2t}{t} - 2\binom{2t-2}{t-1} + 4\binom{2t-6}{t-3} \pmod{8}. \tag{3.5}$$

Proof. The formula is directly verified for $t = 0,1,2$ and 3, so we take $t \geq 4$, in which case the last term on the right hand side is always divisible by 8 (see(3.1)) and may as well be dropped.

Introduce the function $F(t) = \prod_{i=0}^{t} (2i+1)$ and note the identity

(3.6) $$\binom{2t}{t} = \frac{2^t}{t!} F(t-1).$$

Writing $\binom{1+2t}{t} = \frac{2t+1}{t+1} \binom{2t}{t}$ and using (3.6) to substitute for the binomial coefficients, we reduce (3.5) to the equivalent form

(3.7) $$\frac{2^t}{(t+1)!} (F(t)-1) - \frac{2^t}{(t-1)!} F(t-2) \equiv 0 \qquad (\text{mod } 8).$$

Since $F(t+2) \equiv -F(t)$ (mod 4), we have

$$F(t) \equiv \begin{cases} 1 & \\ & (\text{mod } 4) \\ -1 & \end{cases} \qquad \begin{array}{l} t = 4h \text{ or } 4h+3 \\ t = 4h+1 \text{ or } 4h+2 \end{array}$$

From this and (3.1), we easily check that (3.7) is true when $t = 0,3$ (mod 4) or when $\alpha(t+1) \geq 3$; for in these cases each individual term of (3.7) is a multiple of 8. The only remaining possibility is when $t = 2^r+1$, in which case $2^t/(t+1)!$ and $2^t/(t-1)!$ are both even, so we may replace $F(t)$, $F(t-2)$ by their mod 4 values in (3.7), reducing it to

(3.8) $$2\,\frac{2^t}{(t+1)!} - \frac{2^t}{(t-1)!} \equiv 0 \quad (\text{mod } 8) \qquad t = 2^r+1.$$

Now (3.8) is true because its left hand side equals $2^t/(t+1)!$ times a multiple of 4. This ends the proof of our theorem.

<u>Theorem</u> (3.9). <u>The value of</u> $2^t\{n+t,n\}$ mod 8 <u>is given by</u>

(3.10) $$2^t\{n+t,n\} \equiv \binom{n+2t}{t} - 2\binom{n+2t-\varepsilon}{t-1} + 4A \qquad (\text{mod } 8),$$

where $\varepsilon = 1$ or 3 according as n is even or odd, and A depends on the mod 4 value of n as follows:

n(4)	1	2	3	4
A	$\binom{n+2t-7}{t-3}$	$\binom{n+2t-3}{t-2} + \binom{n+2t-6}{t-2}$	$\binom{n+2t-3}{t-2} + \binom{n+2t-7}{t-2}$	0

Proof. By induction on n. When n = 1 this is theorem (3.4). Supposing inductively that the theorem holds for n = 4m+1, we shall derive in succession that it also holds for 4m+2, 4m+3, 4m+4 and 4m+5. The argument is given only for passing from 4m+1 to 4m+2, since the other steps are entirely similar, and equally tedious.

Take, then, n = 4m+1, and assume it has been proved that

$$2^t\{n+t,n\} \equiv \binom{n+2t}{t} - 2\binom{n+2t-3}{t-1} + 4\binom{n+2t-7}{t-3} \pmod 8.$$

By formula (3.3) we have

$$(3.11) \qquad 2^t\{n+1+t,\ n+1\} = \sum_{i=0}^{t} 2^i\{n+i,n\}2^{t-i}\{1+t-i,1\} \equiv$$

$$\sum_{i=0}^{t} \left[\binom{n+2i}{i} - 2\binom{n+2i-3}{i-1} + 4\binom{n+2i-7}{i-3}\right]\left[\binom{1+2t-2i}{t-i} - 2\binom{2t-2i-2}{t-i-1} + 4\binom{2t-2i-6}{t-i-3}\right].$$

We can expand the product in the above expression, drop all multiples of 8, and end up with 6 summation terms. Each term can be written in the form $\Sigma 2^k\rho_k$ according to (2.1). For example,

$$\sum_{i=0}^{t}\binom{n+2i}{i}\binom{1+2t-2i}{t-1} = \sum_{k=0}^{t} 2^k\binom{n+1+2t-k}{t-k} = \binom{n+1+2t}{t} + 2\binom{n+2t}{t-1} + 4\binom{n+2t-1}{t-2} + \cdots$$

If we do this for all the six terms, and ignore multiples of 8 along the way, we end up with the following huge sum

$$(3.12)\quad \{\binom{n+2t+1}{t} + 2\binom{n+2t}{t-1} + 4\binom{n+2t-1}{t-2}\} - 2\{\binom{n+2t-2}{t-1} + 2\binom{n+2t-3}{t-2}\}$$

$$+ 4\binom{n+2t-6}{t-3} - 2\{\binom{n+2t-2}{t-1} + 2\binom{n+2t-3}{t-2}\} + 4\binom{n+2t-5}{t-2} + 4\binom{n+2t-6}{t-3}.$$

In this sum, the second and the third { } can be combined and simplified mod 8, the last term cancels a previous term mod 8. Using the identity $\binom{a}{b} = \binom{a-1}{b} + \binom{a-1}{b-1}$, we get the result

$$\binom{n+1+2t}{t} - 2\binom{n+1+2t-1}{t-1} + 4[\binom{n+1+2t-3}{t-2} + \binom{n+1+2t-6}{t-2}],$$

which is the mod 8 value of $2^t\{n+1+t,n+1\}$, as is to be proved. This establishes the theorem for $n+1 = 4m+2$, and we can go on to $4m+j$, $3 \le j \le 5$ by the same steps of argument, completing the induction.

Of course, if we are only interested in the mod 4 or mod 2 values of $2^t\{n+t,n\}$, things will be greatly simplified. We shall just record

<u>Corollary</u> (3.13). $2^t\{n+k,n\} \equiv \binom{n+2t}{t}$ (mod 2).

<u>Corollary</u> (3.14). $2^t\{n+t,n\} \equiv \binom{n+2t}{t} - 2\binom{n+2t-\varepsilon}{t-1}$ (mod 4).

Now, as a check to the correctness of theorem (3.9), we propose to show, in a somewhat different way.

<u>Proposition</u> (3.15). <u>If</u> $n \equiv 0 \pmod{2^{\ell-1}}$, <u>then</u>

$$2^t\{n+t,n\} \equiv \binom{n+2t}{t} - 2\binom{n+2t-1}{t-1} \pmod{2^{\ell}}.$$

<u>Proof</u>. By induction on ℓ. If $\ell = 1$, see corollary (3.13). Assume the proposition true for n, we will show its truth for 2n. For this purpose write the inductive hypothesis as

$$\left(\frac{e^{2\omega}-1}{2\omega}\right)^n \equiv \sum_{t=0}^{\infty} \left[\binom{n+2t}{t} - 2\binom{n+2t-1}{t-1} \right] \omega^t \qquad (\text{mod } 2^{\ell}).$$

Now, if $a \equiv b$, mod 2^{ℓ}, $\ell > 0$, then $a^2 \equiv b^2$ mod $2^{\ell+1}$ because $a^2 - b^2 = (a-b)^2 + 2b(a-b)$. This is true even if a,b are power series. Hence we get

$$\left(\frac{e^{2\omega}-1}{2\omega}\right)^{2n} \equiv \left(\sum_{t=0}^{\infty} \left[\binom{n+2t}{t} - 2\binom{n+2t-1}{t-1} \right] \omega^t\right)^2 \qquad (\text{mod } 2^{\ell+1}).$$

The coefficient of ω^t in the right hand side is

$$\sum_{i=0}^{t} \left[\binom{n+2i}{i} - 2\binom{n+2i-1}{i-1} \right] \left[\binom{n+2t-2i}{t-i} - 2\binom{n+2t-2i-1}{t-i-1} \right].$$

If we treat this summation in the same way as we treated (3.11), by applying (2.1), and dropping multiples of $2^{\ell+1}$ along the way, we obtain the simple result

$$\binom{2n+2t}{t} - 2\binom{2n+2t-1}{t-1}$$

which is congruent mod $2^{\ell+1}$ to $2^t\{2n+t,2n\}$, the coefficient of ω^t in the left hand side. Thus our induction is completed.

4. Evaluation of the Maunder operations on CP^N

Now we go back to the evaluation on ε of the primary, secondary and tertiary operations, associated with the chain complex $C(3,r)$. In Maunder's notations these are: $\Phi^{1,0}(\varepsilon)$, $\Phi^{2,0}(\varepsilon)$ and $\Phi^{3,0}(\varepsilon)$, where ε is the A-map (1.9) of degree $2n$.

If λ is of degree q, we recall that $\Phi^{K,0}(\lambda)$ is an equivalence class of maps $\xi : C_s \longrightarrow H^*(CP^N;Z_2)$ of degree $q-K+1$, where two maps are considered equivalent if they differ by a map in the indeterminacy of $\Phi^{K,0}$. Also, $\Phi^{K,0}(\lambda)$ is defined only for those A-maps λ such that each $\Phi^{t,0}(\lambda)$, for $K > t > 0$, is the equivalence class containing the zero map (see [13],[14]). For our particular case of the first three operations on CP^N, we will try to present these facts in a simpler equivalent form.

The primary operation is $\Phi^{1,0}(\varepsilon) = \varepsilon d_1$ and from (1.2), it follows that this operation is given by

$$(4.1)\qquad \begin{cases} (\chi Sq^{2r})w^n, \\ Sq^{01}w^n = 0, \\ Sq^1 w^n = 0. \end{cases}$$

Consequently, we can regard $\Phi^{1,0}(\varepsilon)$ as represented only by $(\chi Sq^{2r})w^n$.

We make the first use of the computation developed in section 3, as follows. From Adams's theorem [2;Th.2] together with (1.10) and the congruence (3.13), we get

$$(4.2)\qquad (\chi Sq^{2r})w^n = \binom{n+2r}{r} w^{n+r},$$

and this completes the evaluation of the primary operation $\Phi^{1,0}(\varepsilon)$.