A SHORT TABLE OF INTEGRALS

COMPILED BY

B. O. PEIRCE

HOLLIS PROFESSOR OF MATHEMATICS AND NATURAL PHILOSOPHY
IN HARVARD UNIVERSITY

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B. O. PEIRCE,

Harvard College, Cambridge.

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I. FUNDAMENTAL FORMS.

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1. $\int a\,dx = ax.$

2. $\int af(x)\,dx = a\int f(x)\,dx.$

3. $\int \frac{dx}{x} = \log x.$

4. $\int x^m\,dx = \frac{x^{m+1}}{m+1}$, when m is different from -1.

5. $\int e^x\,dx = e^x.$

6. $\int a^x \log a\,dx = a^x.$

7. $\int \frac{dx}{1+x^2} = \tan^{-1} x.$

8. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x.$

9. $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x.$

10. $\int \frac{dx}{\sqrt{2x-x^2}} = \text{versin}^{-1} x.$

11. $\int \cos x\,dx = \sin x$

12. $\int \sin x\,dx = -\cos x$

13. $\int \operatorname{ctn} x\, dx = \log \sin x.$

14. $\int \tan x\, dx = -\log \cos x.$

15. $\int \tan x \sec x\, dx = \sec x.$

16. $\int \sec^2 x\, dx = \tan x.$

17. $\int \csc^2 x\, dx = -\operatorname{ctn} x.$

In the following formulas, u, v, w, and y represent any functions of x.

18. $\int (u + v + w + \text{etc.})\, dx = \int u dx + \int v dx + \int w dx + \text{etc.}$

19*a*. $\int u dv = uv - \int v du.$

19*b*. $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx.$

20. $\int f(y)\, dx = \int \frac{f(y)\, dy}{\frac{dy}{dx}}.$

II. RATIONAL ALGEBRAIC FUNCTIONS.

A. — Expressions Involving $(a+bx)$.

The substitution of y or z for x, where $y = xz = a + bx$, gives

21. $\int (a+bx)^m dx = \frac{1}{b}\int y^m dy.$

22. $\int x(a+bx)^m dx = \frac{1}{b^2}\int y^m (y-a) dy.$

23. $\int x^n (a+bx)^m dx = \frac{1}{b^{n+1}}\int y^m (y-a)^n dy.$

24. $\int \frac{x^n dx}{(a+bx)^m} = \frac{1}{b^{n+1}}\int \frac{(y-a)^n dy}{y^m}.$

25. $\int \frac{dx}{x^n (a+bx)^m} = -\frac{1}{a^{m+n-1}}\int \frac{(z-b)^{m+n-2} dz}{z^m}.$

Whence

26. $\int \frac{dx}{a+bx} = \frac{1}{b}\log(a+bx).$

27. $\int \frac{dx}{(a+bx)^2} = -\frac{1}{b(a+bx)}.$

28. $\int \frac{dx}{(a+bx)^3} = -\frac{1}{2b(a+bx)^2}.$

29. $\int \frac{x\,dx}{a+bx} = \frac{1}{b^2}[a+bx-a\log(a+bx)].$

30. $\int \frac{x\,dx}{(a+bx)^2} = \frac{1}{b^2}\left[\log(a+bx) + \frac{a}{a+bx}\right].$

31. $\int \frac{x\,dx}{(a+bx)^3} = \frac{1}{b^2}\left[-\frac{1}{a+bx} + \frac{a}{2(a+bx)^2}\right].$

32. $\int \frac{x^2\,dx}{a+bx} = \frac{1}{b^3}\left[\tfrac{1}{2}(a+bx)^2 - 2a(a+bx) + a^2\log(a+bx)\right].$

33. $\int \frac{x^2\,dx}{(a+bx)^2} = \frac{1}{b^3}\left[a+bx - 2a\log(a+bx) - \frac{a^2}{a+bx}\right].$

34. $\int \frac{dx}{x(a+bx)} = -\frac{1}{a}\log\frac{a+bx}{x}.$

35. $\int \frac{dx}{x(a+bx)^2} = \frac{1}{a(a+bx)} - \frac{1}{a^2}\log\frac{a+bx}{x}.$

36. $\int \frac{dx}{x^2(a+bx)} = -\frac{1}{ax} + \frac{b}{a^2}\log\frac{a+bx}{x}.$

B. — Expressions Involving $(a+bx^n)$.

37. $\int \frac{dx}{c^2+x^2} = \frac{1}{c}\tan^{-1}\frac{x}{c}.$

38. $\int \frac{dx}{c^2-x^2} = \frac{1}{2c}\log\frac{c+x}{c-x}.$

39. $\int \frac{dx}{a+bx^2} = \frac{1}{\sqrt{ab}}\tan^{-1}x\sqrt{\frac{b}{a}},$ if $a>0,\ b>0.$

40. $\int \frac{dx}{a+bx^2} = \frac{1}{2\sqrt{-ab}}\log\frac{\sqrt{a}+x\sqrt{-b}}{\sqrt{a}-x\sqrt{-b}},$ if $a>0,\ b<0.$

41. $\int \frac{dx}{(a+bx^2)^2} = \frac{x}{2a(a+bx^2)} + \frac{1}{2a}\int\frac{dx}{a+bx^2}.$

42. $\int \frac{dx}{(a+bx^2)^{m+1}} = \frac{1}{2ma}\,\frac{x}{(a+bx^2)^m} + \frac{2m-1}{2ma}\int\frac{dx}{(a+bx^2)^m}.$

43. $\int \frac{x\,dx}{a+bx^2} = \frac{1}{2b}\log\left(x^2+\frac{a}{b}\right).$

44. $\int \frac{x\,dx}{(a+bx^2)^{m+1}} = \frac{1}{2}\int \frac{dz}{(a+bz)^{m+1}}$, where $z = x^2$.

45. $\int \frac{dx}{x(a+bx^2)} = \frac{1}{2a}\log \frac{x^2}{a+bx^2}$.

46. $\int \frac{x^2\,dx}{a+bx^2} = \frac{x}{b} - \frac{a}{b}\int \frac{dx}{a+bx^2}$.

47. $\int \frac{dx}{x^2(a+bx^2)} = -\frac{1}{ax} - \frac{b}{a}\int \frac{dx}{a+bx^2}$.

48. $\int \frac{x^2\,dx}{(a+bx^2)^{m+1}} = \frac{-x}{2mb(a+bx^2)^m} + \frac{1}{2mb}\int \frac{dx}{(a+bx^2)^m}$.

49. $\int \frac{dx}{x^2(a+bx^2)^{m+1}} = \frac{1}{a}\int \frac{dx}{x^2(a+bx^2)^m} - \frac{b}{a}\int \frac{dx}{(a+bx^2)^{m+1}}$.

50. $\int \frac{dx}{a+bx^3} = \frac{k}{3a}\left[\frac{1}{2}\log\left(\frac{(k+x)^2}{k^2-kx+x^2}\right) + \sqrt{3}\tan^{-1}\frac{2x-k}{k\sqrt{3}}\right]$,

where $bk^3 = a$.

51. $\int \frac{x\,dx}{a+bx^3} = \frac{1}{3bk}\left[\frac{1}{2}\log\left(\frac{k^2-kx+x^2}{(k+x)^2}\right) + \sqrt{3}\tan^{-1}\frac{2x-k}{k\sqrt{3}}\right]$,

where $bk^3 = a$.

52. $\int \frac{dx}{x(a+bx^n)} = \frac{1}{an}\log \frac{x^n}{a+bx^n}$.

53. $\int \frac{dx}{(a+bx^n)^{m+1}} = \frac{1}{a}\int \frac{dx}{(a+bx^n)^m} - \frac{b}{a}\int \frac{x^n\,dx}{(a+bx^n)^{m+1}}$.

54. $\int \frac{x^m\,dx}{(a+bx^n)^{p+1}} = \frac{1}{b}\int \frac{x^{m-n}}{(a+bx^n)^p} - \frac{a}{b}\int \frac{x^{m-n}\,dx}{(a+bx^n)^{p+1}}$.

55. $\int \frac{dx}{x^m(a+bx^n)^{p+1}} = \frac{1}{a}\int \frac{dx}{x^m(a+bx^n)^p} - \frac{b}{a}\int \frac{dx}{x^{m-n}(a+bx^n)^{p+1}}$

56. $$\int x^{m-1}(a+bx^n)^p\,dx = \begin{cases} \dfrac{1}{b(m+np)}\left[x^{m-n}(a+bx^n)^{p+1}-(m-n)a\displaystyle\int x^{m-n-1}(a+bx^n)^p\,dx\right]. \\ \dfrac{1}{m+np}\left[x^m(a+bx^n)^p+npa\displaystyle\int x^{m-1}(a+bx^n)^{p-1}\,dx\right]. \\ \dfrac{1}{ma}\left[x^m(a+bx^n)^{p+1}-(m+np+n)b\displaystyle\int x^{m+n-1}(a+bx^n)^p\,dx\right]. \\ \dfrac{1}{an(p+1)}\left[-x^m(a+bx^n)^{p+1}+(m+np+n)\displaystyle\int x^{m-1}(a+bx^n)^{p+1}\,dx\right]. \end{cases}$$

C. — Expressions Involving $(a+bx+cx^2)$.

Let $X = a+bx+cx^2$ and $q = 4ac-b^2$, then

57. $$\int\frac{dx}{X}=\frac{2}{\sqrt{q}}\tan^{-1}\frac{2cx+b}{\sqrt{q}},\text{ when } q>0.$$

58. $$\int\frac{dx}{X}=\frac{1}{\sqrt{-q}}\log\frac{2cx+b-\sqrt{-q}}{2cx+b+\sqrt{-q}},\text{ when } q<0.$$

59. $$\int\frac{dx}{X^2}=\frac{2cx+b}{qX}+\frac{2c}{q}\int\frac{dx}{X}.$$

60. $$\int\frac{dx}{X^3}=\frac{2cx+b}{q}\left(\frac{1}{2X^2}+\frac{3c}{qX}\right)+\frac{6c^2}{q^2}\int\frac{dx}{X}.$$

61. $$\int\frac{dx}{X^{n+1}}=\frac{2cx+b}{nqX^n}+\frac{2(2n-1)c}{qn}\int\frac{dx}{X^n}.$$

62. $$\int\frac{x\,dx}{X}=\frac{1}{2c}\log X-\frac{b}{2c}\int\frac{dx}{X}.$$

63. $\int \frac{x\,dx}{X^2} = -\frac{bx+2a}{qX} - \frac{b}{q}\int \frac{dx}{X}.$

64. $\int \frac{x\,dx}{X^{n+1}} = -\frac{2a+bx}{nqX^n} - \frac{b(2n-1)}{nq}\int \frac{dx}{X^n}.$

65. $\int \frac{x^2}{X}dx = \frac{x}{c} - \frac{b}{2c^2}\log X + \frac{b^2-2ac}{2c^2}\int \frac{dx}{X}.$

66. $\int \frac{x^2}{X^2}dx = \frac{(b^2-2ac)x+ab}{cqX} + \frac{2a}{q}\int \frac{dx}{X}.$

67. $\int \frac{x^m\,dx}{X^{n+1}} = -\frac{x^{m-1}}{(2n-m+1)cX^n} - \frac{n-m+1}{2n-m+1}\cdot\frac{b}{c}\int \frac{x^{m-1}\,dx}{X^{n+1}}$

$$+\frac{m-1}{2n-m+1}\cdot\frac{a}{c}\int \frac{x^{m-2}\,dx}{X^{n+1}}.$$

68. $\int \frac{dx}{xX} = \frac{1}{2a}\log\frac{x^2}{X} - \frac{b}{2a}\int \frac{dx}{X}.$

69. $\int \frac{dx}{x^2X} = \frac{b}{2a^2}\log\frac{X}{x^2} - \frac{1}{ax} + \left(\frac{b^2}{2a^2} - \frac{c}{a}\right)\int \frac{dx}{X}.$

70. $\int \frac{dx}{x^mX^{n+1}} = -\frac{1}{(m-1)ax^{m-1}X^n} - \frac{n+m-1}{m-1}\cdot\frac{b}{a}\int \frac{dx}{x^{m-1}X^{n+1}}$

$$-\frac{2n+m-1}{m-1}\cdot\frac{c}{a}\int \frac{dx}{x^{m-2}X^{n+1}}.$$

D. — Rational Fractions.

Every proper fraction can be represented by the general form:

$$\frac{f(x)}{F(x)} = \frac{g_1x^{n-1}+g_2x^{n-2}+g_3x^{n-3}+\cdots+g_n}{x^n+k_1x^{n-1}+k_2x^{n-2}+\cdots+k_n}.$$

a, b, c, etc., are the roots of the equation $F(x)=0$, so that

$$F(x) = (x-a)^p\,(x-b)^q\,(x-c)^r\cdots.$$

then
$$\frac{f(x)}{F(x)}=\frac{A_1}{(x-a)^p}+\frac{A_2}{(x-a)^{p-1}}+\frac{A_3}{(x-a)^{p-2}}+\cdots+\frac{A_p}{x-a}$$
$$+\frac{B_1}{(x-b)^q}+\frac{B_2}{(x-b)^{q-1}}+\frac{B_3}{(x-b)^{q-2}}+\cdots+\frac{B_q}{x-b}$$
$$+\frac{C_1}{(x-c)^r}+\frac{C_2}{(x-c)^{r-1}}+\frac{C_3}{(x-c)^{r-2}}+\cdots+\frac{C_r}{x-c}$$
$$+\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

Where the numerators of the separate fractions may be determined by the equations
$$A_m=\frac{\phi_1^{[m-1]}(a)}{(m-1)\,!},\quad B_m=\frac{\phi_2^{[m-1]}(b)}{(m-1)\,!},\ \text{etc., etc.}$$
$$\phi_1(x)=\frac{f(x)\,(x-a)^p}{F(x)},\quad \phi_2(x)=\frac{f(x)\,(x-b)^q}{F(x)},\ \text{etc., etc.}$$

If a, b, c, etc., are single roots, then $p=q=r=\cdots=1$, and
$$\frac{f(x)}{F(x)}=\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}\cdots$$
where
$$A=\frac{f(a)}{F'(a)},\quad B=\frac{f(b)}{F'(b)},\ \text{etc.}$$

The simpler fractions, into which the original fraction is thus divided, may be integrated by means of the following formulas:

71. $$\int\frac{h\,dx}{(mx+n)^l}=\int\frac{h\,d(mx+n)}{m\,(mx+n)^l}=\frac{h}{m\,(1-l)\,(mx+n)^{l-1}}.$$

72. $$\int\frac{h\,dx}{mx+n}=\frac{h}{m}\log\,(mx+n).$$

If any of the roots of the equation $f(x)=0$ are imaginary, the parts of the integral which arise from conjugate roots can be combined together and the integral brought into a real form. The following formula, in which $i=\sqrt{-1}$, is often useful in combining logarithms of conjugate complex quantities:

73. $$\log\,(x\pm yi)=\tfrac{1}{2}\log\,(x^2+y^2)\pm i\tan^{-1}\frac{y}{x}.$$

III. IRRATIONAL ALGEBRAIC FUNCTIONS.

A.—Expressions Involving $\sqrt{a+bx}$.

The substitution of a new variable of integration, $y=\sqrt{a+bx}$, gives

74. $\int\sqrt{a+bx}\,dx=\frac{2}{3b}\sqrt{(a+bx)^3}.$

75. $\int x\sqrt{a+bx}\,dx=-\frac{2(2a-3bx)\sqrt{(a+bx)^3}}{15b^2}.$

76. $\int x^2\sqrt{a+bx}\,dx=\frac{2(8a^2-12abx+15b^2x^2)\sqrt{(a+bx)^3}}{105b^3}.$

77. $\int\frac{\sqrt{a+bx}}{x}dx=2\sqrt{a+bx}+a\int\frac{dx}{x\sqrt{a+bx}}.$

78. $\int\frac{dx}{\sqrt{a+bx}}=\frac{2\sqrt{a+bx}}{b}.$

79. $\int\frac{x\,dx}{\sqrt{a+bx}}=-\frac{2(2a-bx)}{3b^2}\sqrt{a+bx}.$

80. $\int\frac{x^2\,dx}{\sqrt{a+bx}}=\frac{2(8a^2-4abx+3b^2x^2)}{15b^3}\sqrt{a+bx}.$

81. $\int\frac{dx}{x\sqrt{a+bx}}=\frac{1}{\sqrt{a}}\log\left(\frac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}}\right)$, for $a>0$.

82. $\int\frac{dx}{x\sqrt{a+bx}}=\frac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\frac{a+bx}{-a}}$, for $a<0$.

83. $\int\frac{dx}{x^2\sqrt{a+bx}}=-\frac{\sqrt{a+bx}}{ax}-\frac{b}{2a}\int\frac{dx}{x\sqrt{a+bx}}.$

84. $\int (a+bx)^{\pm\frac{n}{2}} dx = \frac{2}{b}\int y^{1\pm n} dy = \frac{2(a+bx)^{\frac{2\pm n}{2}}}{b(2\pm n)}.$

85. $\int x(a+bx)^{\pm\frac{n}{2}} dx = \frac{2}{b^2}\left[\frac{(a+bx)^{\frac{4\pm n}{2}}}{4\pm n} - \frac{a(a+bx)^{\frac{2\pm n}{2}}}{2\pm n}\right].$

86. $\int \frac{x^m\,dx}{\sqrt{a+bx}} = \frac{2x^m\sqrt{a+bx}}{(2m+1)b} - \frac{2ma}{(2m+1)b}\int\frac{x^{m-1}\,dx}{\sqrt{a+bx}}.$

87. $\int \frac{dx}{x^n\sqrt{a+bx}} = -\frac{\sqrt{a+bx}}{(n-1)ax^{n-1}} - \frac{(2n-3)b}{(2n-2)a}\int\frac{dx}{x^{n-1}\sqrt{a+bx}}.$

88. $\int \frac{(a+bx)^{\frac{n}{2}}\,dx}{x} = b\int(a+bx)^{\frac{n-2}{2}}dx + a\int\frac{(a+bx)^{\frac{n-2}{2}}}{x}dx.$

89. $\int \frac{dx}{x(a+bx)^{\frac{m}{2}}} = \frac{1}{a}\int\frac{dx}{x(a+bx)^{\frac{m-2}{2}}} - \frac{b}{a}\int\frac{dx}{(a+bx)^{\frac{m}{2}}}.$

B. — Expressions* Involving $\sqrt{x^2 \pm a^2}$ and $\sqrt{a^2 - x^2}$.

90. $\int \sqrt{x^2\pm a^2}\,dx = \frac{1}{2}\left[x\sqrt{x^2\pm a^2} \pm a^2\log\left(x+\sqrt{x^2\pm a^2}\right)\right].$

91. $\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}\left(x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a}\right).$

92. $\int \frac{dx}{\sqrt{x^2\pm a^2}} = \log\left(x+\sqrt{x^2\pm a^2}\right).$

93. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}.$

94. $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\cos^{-1}\frac{a}{x}.$

95. $\int \frac{dx}{x\sqrt{a^2\pm x^2}} = -\frac{1}{a}\log\left(\frac{a+\sqrt{a^2\pm x^2}}{x}\right).$

96. $\int \frac{\sqrt{a^2\pm x^2}}{x}dx = \sqrt{a^2\pm x^2} - a\log\frac{a+\sqrt{a^2\pm x^2}}{x}.$

* These equations are all special cases of more general equations given in the next section.

97. $\int \frac{\sqrt{x^2-a^2}}{x}\,dx = \sqrt{x^2-a^2} - a\cos^{-1}\frac{a}{x}.$

98. $\int \frac{x\,dx}{\sqrt{a^2 \pm x^2}} = \pm\sqrt{a^2 \pm x^2}.$

99. $\int \frac{x\,dx}{\sqrt{x^2-a^2}} = \sqrt{x^2-a^2}.$

100. $\int x\sqrt{x^2 \pm a^2}\,dx = \frac{1}{3}\sqrt{(x^2 \pm a^2)^3}.$

101. $\int x\sqrt{a^2-x^2}\,dx = -\frac{1}{3}\sqrt{(a^2-x^2)^3}.$

102. $\int \sqrt{(x^2 \pm a^2)^3}\,dx$

$$= \frac{1}{4}\left[x\sqrt{(x^2 \pm a^2)^3} \pm \frac{3a^2x}{2}\sqrt{x^2 \pm a^2} + \frac{3a^4}{2}\log\left(x+\sqrt{x^2 \pm a^2}\right)\right].$$

103. $\int \sqrt{(a^2-x^2)^3}\,dx$

$$= \frac{1}{4}\left[x\sqrt{(a^2-x^2)^3} + \frac{3a^2x}{2}\sqrt{a^2-x^2} + \frac{3a^4}{2}\sin^{-1}\frac{x}{a}\right].$$

104. $\int \frac{dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{\pm x}{a^2\sqrt{x^2 \pm a^2}}.$

105. $\int \frac{dx}{\sqrt{(a^2-x^2)^3}} = \frac{x}{a^2\sqrt{a^2-x^2}}.$

106. $\int \frac{x\,dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{-1}{\sqrt{x^2 \pm a^2}}.$

107. $\int \frac{x\,dx}{\sqrt{(a^2-x^2)^3}} = \frac{1}{\sqrt{a^2-x^2}}.$

108. $\int x\sqrt{(x^2 \pm a^2)^3}\,dx = \frac{1}{5}\sqrt{(x^2 \pm a^2)^5}.$

109. $\int x\sqrt{(a^2-x^2)^3}\,dx = -\frac{1}{5}\sqrt{(a^2-x^2)^5}.$

110. $\int x^2\sqrt{x^2 \pm a^2}\,dx$

$$= \frac{x}{4}\sqrt{(x^2 \pm a^2)^3} \mp \frac{a^2}{8}\left(x\sqrt{x^2 \pm a^2} \pm a^2 \log\left(x + \sqrt{x^2 \pm a^2}\right)\right).$$

111. $\int x^2\sqrt{a^2 - x^2}\,dx$

$$= -\frac{x}{4}\sqrt{(a^2 - x^2)^3} + \frac{a^2}{8}\left(x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\frac{x}{a}\right).$$

112. $\int \frac{x^2\,dx}{\sqrt{x^2 \pm a^2}} = \frac{x}{2}\sqrt{x^2 \pm a^2} \mp \frac{a^2}{2}\log\left(x + \sqrt{x^2 \pm a^2}\right).$

113. $\int \frac{x^2\,dx}{\sqrt{a^2 - x^2}} = -\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}.$

114. $\int \frac{dx}{x^2\sqrt{x^2 \pm a^2}} = \mp\frac{\sqrt{x^2 \pm a^2}}{a^2 x}.$

115. $\int \frac{dx}{x^2\sqrt{a^2 - x^2}} = -\frac{\sqrt{a^2 - x^2}}{a^2 x}.$

116. $\int \frac{\sqrt{x^2 \pm a^2}\,dx}{x^2} = -\frac{\sqrt{x^2 \pm a^2}}{x} + \log\left(x + \sqrt{x^2 \pm a^2}\right).$

117. $\int \frac{\sqrt{a^2 - x^2}}{x^2}\,dx = -\frac{\sqrt{a^2 - x^2}}{x} - \sin^{-1}\frac{x}{a}.$

118. $\int \frac{x^2\,dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{-x}{\sqrt{x^2 \pm a^2}} + \log\left(x + \sqrt{x^2 \pm a^2}\right).$

119. $\int \frac{x^2\,dx}{\sqrt{(a^2 - x^2)^3}} = \frac{x}{\sqrt{a^2 - x^2}} - \sin^{-1}\frac{x}{a}.$

C. — Expressions Involving $\sqrt{a + bx + cx^2}$.

Let $X = a + bx + cx^2$, $q = 4ac - b^2$, and $k = \frac{4c}{q}$. In order to rationalize the function $f(x, \sqrt{a + bx + cx^2})$ we may put $\sqrt{a + bx + cx^2} = \sqrt{\pm c}\sqrt{A + Bx \pm x^2}$, according as c is positive or negative, and then substitute for x a new variable z, such that

$z = \sqrt{A + Bx + x^2} - x$, if $c > 0$.

$z = \frac{\sqrt{A + Bx - x^2} - \sqrt{A}}{x}$, if $c < 0$ and $\frac{a}{-c} > 0$.

$z = \sqrt{\frac{x - \beta}{a - x}}$, where a and β are the roots of the equation

$A + Bx - x^2 = 0$, if $c < 0$ and $\frac{a}{-c} < 0$.

By rationalization, or by the aid of reduction formulas, may be obtained the values of the following integrals:

120. $\int \frac{dx}{\sqrt{X}} = \frac{1}{\sqrt{c}} \log\left(\sqrt{X} + x\sqrt{c} + \frac{b}{2\sqrt{c}}\right)$, if $c > 0$.

121. $\int \frac{dx}{\sqrt{X}} = \frac{1}{\sqrt{-c}} \sin^{-1}\left(\frac{-2cx - b}{\sqrt{b^2 - 4ac}}\right)$, if $c < 0$.

122. $\int \frac{dx}{X\sqrt{X}} = \frac{2(2cx + b)}{q\sqrt{X}}$.

123. $\int \frac{dx}{X^2\sqrt{X}} = \frac{2(2cx + b)}{3q\sqrt{X}}\left(\frac{1}{X} + 2k\right)$.

124. $\int \frac{dx}{X^n\sqrt{X}} = \frac{2(2cx+b)\sqrt{X}}{(2n-1)qX^n} + \frac{2k(n-1)}{2n-1}\int \frac{dx}{X^{n-1}\sqrt{X}}$.

125. $\int \sqrt{X}\,dx = \frac{(2cx + b)\sqrt{X}}{4c} + \frac{1}{2k}\int \frac{dx}{\sqrt{X}}$.

126. $\int X\sqrt{X}\,dx = \frac{(2cx + b)\sqrt{X}}{8c}\left(X + \frac{3}{2k}\right) + \frac{3}{8k^2}\int \frac{dx}{\sqrt{X}}$.

127. $\int X^2\sqrt{X}\,dx = \frac{(2cx+b)\sqrt{X}}{12c}\left(X^2 + \frac{5X}{4k} + \frac{15}{8k^2}\right) + \frac{5}{16k^3}\int \frac{dx}{\sqrt{X}}$

128. $\int X^n\sqrt{X}\,dx = \frac{(2cx + b)X^n\sqrt{X}}{4(n+1)c} + \frac{2n+1}{2(n+1)k}\int \frac{X^n\,dx}{\sqrt{X}}$.

129. $\int \frac{x\,dx}{\sqrt{X}} = \frac{\sqrt{X}}{c} - \frac{b}{2c}\int \frac{dx}{\sqrt{X}}$.

130. $\int \frac{x\,dx}{X\sqrt{X}} = -\frac{2(bx+2a)}{q\sqrt{X}}.$

131. $\int \frac{x\,dx}{X^n\sqrt{X}} = -\frac{\sqrt{X}}{(2n-1)cX^n} - \frac{b}{2c}\int \frac{dx}{X^n\sqrt{X}}.$

132. $\int \frac{x^2\,dx}{\sqrt{X}} = \left(\frac{x}{2c} - \frac{3b}{4c^2}\right)\sqrt{X} + \frac{3b^2-4ac}{8c^2}\int \frac{dx}{\sqrt{X}}.$

133. $\int \frac{x^2\,dx}{X\sqrt{X}} = \frac{(2b^2-4ac)x+2ab}{cq\sqrt{X}} + \frac{1}{c}\int \frac{dx}{\sqrt{X}}.$

134. $\int \frac{x^2\,dx}{X^n\sqrt{X}} = \frac{(2b^2-4ac)x+2ab}{(2n-1)cqX^{n-1}\sqrt{X}} + \frac{4ac+(2n-3)b^2}{(2n-1)cq}\int \frac{dx}{X^{n-1}\sqrt{X}}$

135. $\int \frac{x^3\,dx}{\sqrt{X}} = \left(\frac{x^2}{3c} - \frac{5bx}{12c^2} + \frac{5b^2}{8c^3} - \frac{2a}{3c^2}\right)\sqrt{X} + \left(\frac{3ab}{4c^2} - \frac{5b^3}{16c^3}\right)\int \frac{dx}{\sqrt{X}}$

136. $\int x\sqrt{X}\,dx = \frac{X\sqrt{X}}{3c} - \frac{b}{2c}\int \sqrt{X}\,dx.$

137. $\int xX\sqrt{X}\,dx = \frac{X^2\sqrt{X}}{5c} - \frac{b}{2c}\int X\sqrt{X}\,dx.$

138. $\int \frac{xX^n\,dx}{\sqrt{X}} = \frac{X^n\sqrt{X}}{(2n+1)c} - \frac{b}{2c}\int \frac{X^n\,dx}{\sqrt{X}}.$

139. $\int x^2\sqrt{X}\,dx = \left(x - \frac{5b}{6c}\right)\frac{X\sqrt{X}}{4c} + \frac{5b^2-4ac}{16c^2}\int \sqrt{X}\,dx.$

140. $\int \frac{x^2X^n\,dx}{\sqrt{X}} = \frac{xX^n\sqrt{X}}{2(n+1)c} - \frac{(2n+3)b}{4(n+1)c}\int \frac{xX^n\,dx}{\sqrt{X}}$

$$- \frac{a}{2(n+1)c}\int \frac{X^n\,dx}{\sqrt{X}}.$$

141. $\int x^3\sqrt{X}\,dx = \left(x^2 - \frac{7bx}{8c} + \frac{35b^2}{48c^2} - \frac{2a}{3c}\right)\frac{X\sqrt{X}}{5c}$

$$+ \left(\frac{3ab}{8c^2} - \frac{7b^3}{32c^3}\right)\int \sqrt{X}\,dx.$$