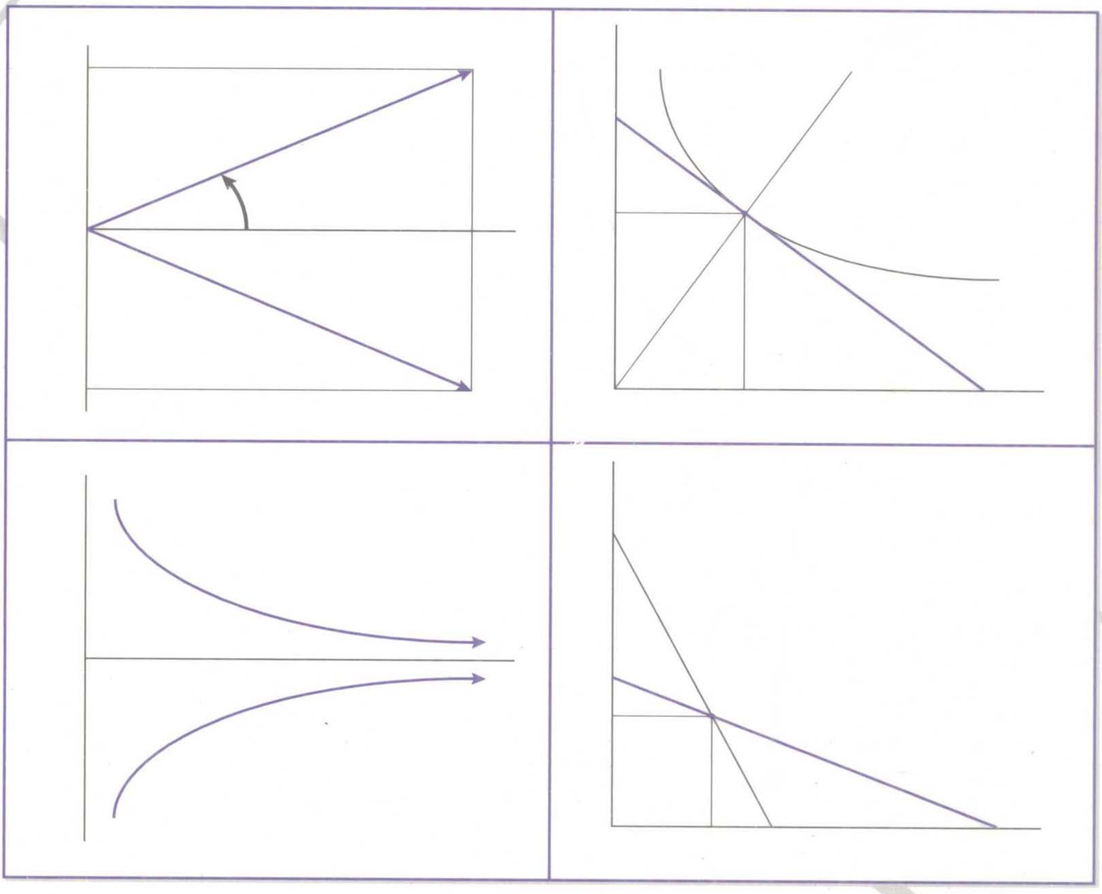
Mathematical Methods and Models for Economists



Angel de la Fuente

MATHEMATICAL METHODS AND MODELS FOR ECONOMISTS

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MATHEMATICAL METHODS AND MODELS FOR ECONOMISTS

This book is intended as a textbook for a first-year Ph.D. course in mathematics for economists and as a reference for graduate students in economics. It provides a self-contained, rigorous treatment of most of the concepts and techniques required to follow the standard first-year theory sequence in microeconomics and macroeconomics. The topics covered include an introduction to analysis in metric spaces, differential calculus, comparative statics, convexity, static optimization, dynamical systems, and dynamic optimization. The book includes a large number of applications to standard economic models and more than two hundred problems that are fully worked out.

Angel de la Fuente is Assistant Professor of Economics at the Instituto de Análisis Económica (CSIC), Adjunct Professor at the Department of Economics of the Universidad Autónoma de Barcelona, and Research Affiliate at the Centre for Economic Policy Research (CEPR), London. Besides his interest in mathematical economics, he specializes in growth and regional economics. Professor de la Fuente has published in the *Journal of Monetary Economics*, the *Journal of Economic Dynamics and Control*, *Economic Policy*, *Revista Española de Economía*, and *Investigaciones Económicas*, among other journals.

A mis padres

Preface and Acknowledgments

Much of the time of the average graduate student in economics is spent learning a new language, that of mathematics. Although the investment does eventually pay off in many ways, the learning process can be quite painful. I know because I have been there. I remember the long nights spent puzzling over the mysteries of the Hamiltonian, the frustration of not understanding a single one of the papers in my second macroeconomics reading list, the culture shock that came with the transition from the undergraduate textbooks, with their familiar diagrams and intuitive explanations, into Debreu's *Theory of Value*, and my despair before the terse and incredibly dry prose of the mathematics texts where I sought enlightenment about the arcane properties of contractions.

This book is an attempt to make the transition into graduate economics somewhat less painful. Although some of my readers may never believe me, I have tried to do a number of things that should make their lives a bit easier. The first has been to collect in one place, with a homogeneous notation, most of the mathematical concepts, results, and techniques that are required to follow the standard first- and second-year theory courses. I have also tried to organize this material into a logical sequence and have illustrated its applications to some of the standard models. And last but not least, I have attempted to provide rigorous proofs for most of the results as a way to get the reader used to formal reasoning. Although a lot of effort has gone into making the text as clear as possible, the result is still far from entertaining. Most students without a good undergraduate background in mathematics are likely to find the going a bit rough at times. They have all my sympathy and the assurance that it does build character.

This book has been long in the making. It started out as a set of notes that I wrote for myself during my first year at Penn. Those notes were then refined for the benefit of my younger classmates when I became a teaching assistant, and they finally grew into lecture notes when I had the misfortune to graduate and was forced onto the other side of the lectern. Along the way,

I have had a lot of help. Much of the core material can be traced back to class lectures by Richard Kihlstrom, George Mailath, Beth Allen, David Cass, Maurice Obstfeld, Allan Drazen, Costas Azariadis, and Randy Wright. The first typed version of these notes was compiled jointly with Francis Bloch over a long and sticky Philadelphia summer as a reference for an introductory summer course for incoming students. Francis had the good sense to jump ship right after that, but some of his material is still here in one form or another. Several colleagues and friends have had the patience to read through various portions of the manuscript and have made many useful comments and suggestions. Among these, I would especially like to thank David Pérez and Maite Naranjo, who has also contributed a couple of the more difficult proofs. Thanks are also due to several generations of students at the Universidad Autónoma de Barcelona and various other places, who, while sweating through successive versions of this manuscript, have helped me to improve it in various ways and have detected a fair number of errors, as well as the usual typos. Finally, I would like to thank Conchi Rodriguez, Tere Lorenz, and the rest of the staff at the Instituto de Análisis Económica for their secretarial support and for their heroic behavior at the Xerox machine.

Contents

Part I

Preliminaries

1

Review of Basic Concepts

This chapter reviews some basic concepts that will be used throughout the text. One of its central themes is that relations and functions can be used to introduce different types of "structures" in sets. Thus relations of a certain type can be used to order sets according to criteria like precedence, size, or goodness; algebraic operations are defined using functions, and a function that formalizes the notion of distance between two elements of a set can be used to define topological concepts like convergence and continuity. In addition, we also introduce some simple notions of logic and discuss several methods of proof that will be used extensively later on.

1. Sets

A *set* is a collection of objects we call elements. We will denote sets by capital letters, and elements by lowercase letters. If x is an element of a set X, we write $x \in X$, and $x \notin X$ otherwise. A set A is a *subset* of X if all elements of A belong to X. This is written $A \subseteq X$ (A is contained in X). Formally, we can write

$$A \subseteq X \Leftrightarrow (x \in A \Rightarrow x \in X)$$

where the one-way arrow ($\Rightarrow$) denotes implication, and the two-way arrow ($\Leftrightarrow$) indicates equivalence. Two sets, A and B, are equal if they have the same elements, that is, $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$. The symbol $\varnothing$ denotes the *empty set*, a set with no elements. By convention, $\varnothing$ is a subset of any set X.

Given a set X, the *power set* of X, written $P(X)$ or 2^X, is the set consisting of all the subsets A of X. A class or *family of sets* in X is a subset of $P(X)$, that is, a set whose elements are subsets of X. We will use "hollow" capital letters to denote families of sets. For example,

$$\mathbb{A} = \{A_i;\ A_i \subseteq X,\ i \in I\}$$

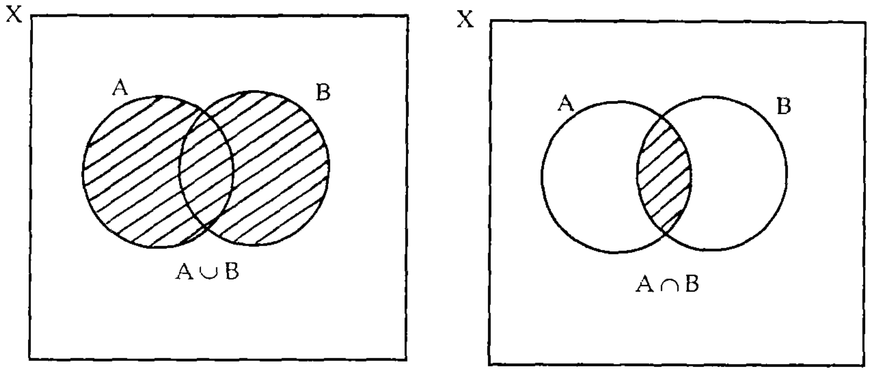

Figure 1.1. Union and intersection of two sets.

where I is some index set, such as the set of all natural numbers smaller than some given number n.

In what follows we will take as given some set X (the universal set), and assuming that "there is nothing" outside X, we will work with its subsets. Given two subsets of X, A and B, we define their *union*, $A \cup B$, as the set

$$A \cup B = \{x \in X;\ x \in A \text{ or } x \in B\}$$

That is, $A \cup B$ is the set of elements of X that belong to A or to B or to both. Similarly, the *intersection* of A and B, denoted by $A \cap B$, is the set whose elements belong to both sets at the same time:

$$A \cap B = \{x \in X;\ x \in A \text{ and } x \in B\}$$

These concepts can be extended in a natural way to classes of more than two sets. Given a family of subsets of X, $\mathbb{A} = \{A_i;\ i \in I\}$, its union and intersection are given respectively by

$$\cup\mathbb{A} = \cup_{i \in I} A_i = \{x \in X;\ \exists i \in I \text{ s.th. } x \in A_i\} \quad \text{and}$$
$$\cap\mathbb{A} = \cap_{i \in I} A_i = \{x \in X;\ x \in A_i\ \forall i \in I\}$$

where the *existential quantifier* "$\exists$" means "there exists some," the *universal quantifier* "$\forall$" means "for all," and "s.th." means "such that." That is, x is an element of the union $\cup\mathbb{A}$ if it belongs to at least one of the sets in the family $\mathbb{A}$, and it is an element of the intersection if it belongs to all sets in the class.

The following theorem summarizes the basic properties of unions and intersections of sets.

Theorem 1.1. Properties of unions and intersections of sets. Let A, B, *and* C *be three subsets of* X. *Then the following properties hold:*