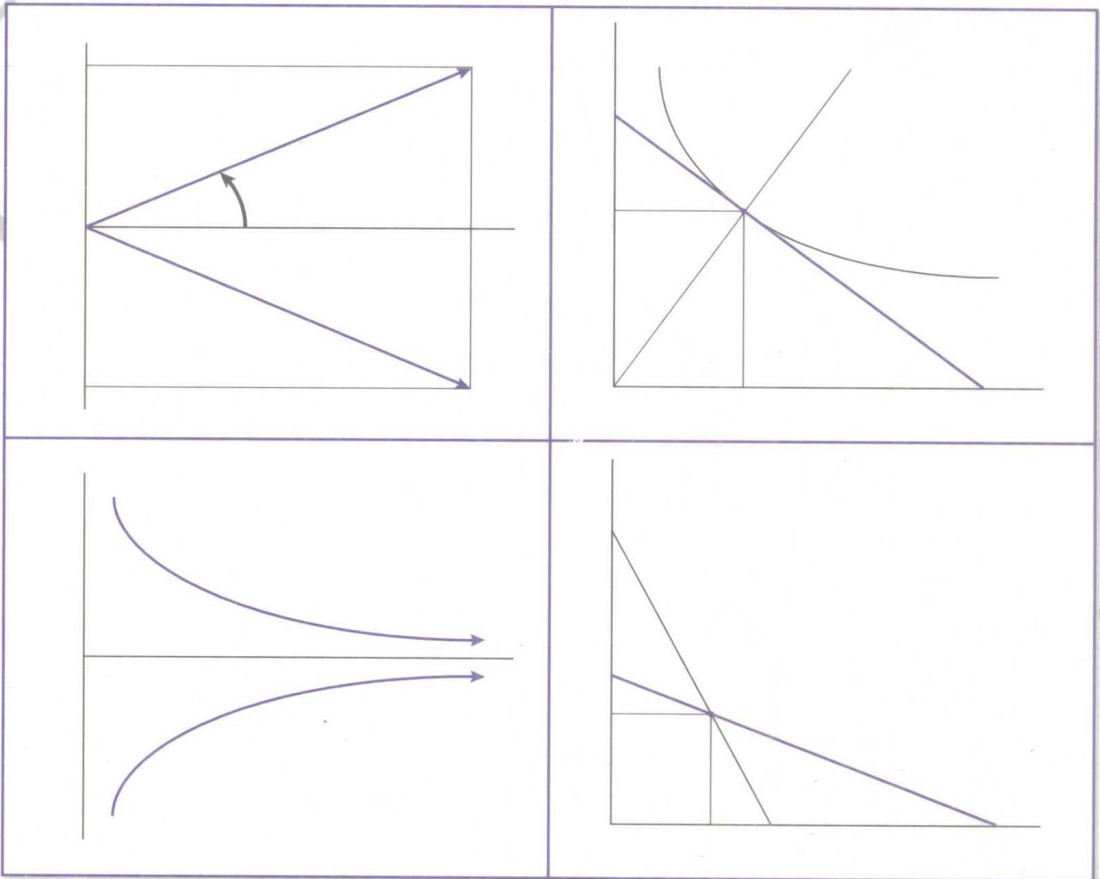


Mathematical Methods and Models for Economists



Angel de la Fuente

MATHEMATICAL METHODS AND MODELS FOR ECONOMISTS

ANGEL DE LA FUENTE

Instituto de Análisis Económica (CSIC), Barcelona



CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK <http://www.cup.cam.ac.uk>
40 West 20th Street, New York, NY 10011-4211, USA <http://www.cup.org>
10 Stamford Road, Oakleigh, Melbourne 3166, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain

© Angel de la Fuente 2000

This book is in copyright. Subject to statutory exception
and to the provisions of relevant collective licensing agreements,
no reproduction of any part may take place without
the written permission of Cambridge University Press.

First published 2000

Printed in the United States of America

Typeface Times 11/14 pt. System QuarkXPress [BTS]

A catalog record for this book is available from the British Library.

Library of Congress Cataloging in Publication Data

Fuente, Angel de la.
Mathematical methods and models for economists / Angel de la Fuente.
p. cm.

Includes index.

ISBN 0-521-58512-0 hb.-ISBN 0-521-58529-5 pb

1. Economics--Statistical methods. 2. Mathematical models.

I. Title.

HB137.F83 1999

330'.01'51--dc21

98-52086
CIP

0 521 58512 0 hardback
0 521 58529 5 paperback

MATHEMATICAL METHODS AND MODELS FOR ECONOMISTS

This book is intended as a textbook for a first-year Ph.D. course in mathematics for economists and as a reference for graduate students in economics. It provides a self-contained, rigorous treatment of most of the concepts and techniques required to follow the standard first-year theory sequence in microeconomics and macroeconomics. The topics covered include an introduction to analysis in metric spaces, differential calculus, comparative statics, convexity, static optimization, dynamical systems, and dynamic optimization. The book includes a large number of applications to standard economic models and more than two hundred problems that are fully worked out.

Angel de la Fuente is Assistant Professor of Economics at the Instituto de Análisis Económica (CSIC), Adjunct Professor at the Department of Economics of the Universidad Autónoma de Barcelona, and Research Affiliate at the Centre for Economic Policy Research (CEPR), London. Besides his interest in mathematical economics, he specializes in growth and regional economics. Professor de la Fuente has published in the *Journal of Monetary Economics*, the *Journal of Economic Dynamics and Control*, *Economic Policy*, *Revista Española de Economía*, and *Investigaciones Económicas*, among other journals.

A mis padres

Preface and Acknowledgments

Much of the time of the average graduate student in economics is spent learning a new language, that of mathematics. Although the investment does eventually pay off in many ways, the learning process can be quite painful. I know because I have been there. I remember the long nights spent puzzling over the mysteries of the Hamiltonian, the frustration of not understanding a single one of the papers in my second macroeconomics reading list, the culture shock that came with the transition from the undergraduate textbooks, with their familiar diagrams and intuitive explanations, into Debreu's *Theory of Value*, and my despair before the terse and incredibly dry prose of the mathematics texts where I sought enlightenment about the arcane properties of contractions.

This book is an attempt to make the transition into graduate economics somewhat less painful. Although some of my readers may never believe me, I have tried to do a number of things that should make their lives a bit easier. The first has been to collect in one place, with a homogeneous notation, most of the mathematical concepts, results, and techniques that are required to follow the standard first- and second-year theory courses. I have also tried to organize this material into a logical sequence and have illustrated its applications to some of the standard models. And last but not least, I have attempted to provide rigorous proofs for most of the results as a way to get the reader used to formal reasoning. Although a lot of effort has gone into making the text as clear as possible, the result is still far from entertaining. Most students without a good undergraduate background in mathematics are likely to find the going a bit rough at times. They have all my sympathy and the assurance that it does build character.

This book has been long in the making. It started out as a set of notes that I wrote for myself during my first year at Penn. Those notes were then refined for the benefit of my younger classmates when I became a teaching assistant, and they finally grew into lecture notes when I had the misfortune to graduate and was forced onto the other side of the lectern. Along the way,

I have had a lot of help. Much of the core material can be traced back to class lectures by Richard Kihlstrom, George Mailath, Beth Allen, David Cass, Maurice Obstfeld, Allan Drazen, Costas Azariadis, and Randy Wright. The first typed version of these notes was compiled jointly with Francis Bloch over a long and sticky Philadelphia summer as a reference for an introductory summer course for incoming students. Francis had the good sense to jump ship right after that, but some of his material is still here in one form or another. Several colleagues and friends have had the patience to read through various portions of the manuscript and have made many useful comments and suggestions. Among these, I would especially like to thank David Pérez and Maite Naranjo, who has also contributed a couple of the more difficult proofs. Thanks are also due to several generations of students at the Universidad Autónoma de Barcelona and various other places, who, while sweating through successive versions of this manuscript, have helped me to improve it in various ways and have detected a fair number of errors, as well as the usual typos. Finally, I would like to thank Conchi Rodriguez, Tere Lorenz, and the rest of the staff at the Instituto de Análisis Económica for their secretarial support and for their heroic behavior at the Xerox machine.

Contents

<i>Preface and Acknowledgments</i>	<i>xi</i>
PART I. PRELIMINARIES	
1. Review of Basic Concepts	3
1. Sets	3
2. A Bit of Logic	6
a. Properties and Quantifiers	6
b. Implication	9
c. Methods of Proof	11
3. Relations	15
a. Equivalence Relations and Decomposition of a Set into Classes	17
b. Order Relations and Ordered Sets	18
4. Functions	20
5. Algebraic Structures	24
a. Groups and Fields	25
b. Vector Spaces	28
6. The Real Number System	29
a. A Set of Axioms for the Real Number System	30
b. The Supremum Property	31
c. Absolute Value	35
7. Complex Numbers	36
Bibliography	37
Notes	38
2. Metric and Normed Spaces	39
1. Metric and Normed Spaces	40
2. Convergence of Sequences in Metric Spaces	46
3. Sequences in \mathbb{R} and \mathbb{R}^m	49
4. Open and Closed Sets	58

a. Interior, Boundary and Closure of a Set	59
b. Limit Points and a Sequential Characterization in Terms of Sequences	61
5. Limits of Functions	64
6. Continuity in Metric Spaces	66
7. Complete Metric Spaces and the Contraction Mapping Theorem	79
a. Cauchy Sequences and Complete Metric Spaces	80
b. Operators and the Contraction Mapping Theorem	85
8. Compactness and the Extreme-Value Theorem	90
a. Compactness and Some Characterizations	90
b. Relation with Other Topological Properties	95
c. Continuous Functions on Compact Sets	98
9. Connected Sets	100
10. Equivalent Metrics and Norms	104
11. Continuity of Correspondences in E^n	108
Bibliography	114
Notes	115
3. Vector Spaces and Linear Transformations	117
1. Linear Independence and Bases	117
2. Linear Transformations	122
a. Image and Kernel of a Linear Function	123
b. The Inverse of a Linear Transformation	126
3. Isomorphisms	127
4. Linear Mappings between Normed Spaces	132
a. Linear Homeomorphisms	134
b. The Norm of a Linear Mapping	135
c. The Normed Vector Space $L(\mathbb{R}^n, \mathbb{R}^m)$	137
5. Change of Basis and Similarity	144
6. Eigenvalues and Eigenvectors	146
Appendix: Polynomial Equations	152
Bibliography	154
Notes	155
4. Differential Calculus	156
1. Differentiable Univariate Real Functions	156
2. Partial and Directional Derivatives	163
3. Differentiability	169
4. Continuous Differentiability	179
5. Homogeneous Functions	187
Bibliography	190
Notes	190

PART II. STATICS

5. Static Models and Comparative Statics	195
1. Linear Models	196
2. Comparative Statics and the Implicit-Function Theorem	200
a. Derivatives of Implicit Functions and Comparative Statics	202
b. The Implicit-Function Theorem	205
3. Existence of Equilibrium	218
a. The Intermediate Value Theorem	219
b. Fixed Point Theorems	221
4. Problems	224
Bibliography	227
Notes	228
6. Convex Sets and Concave Functions	229
1. Convex Sets and Separation Theorems in \mathbb{R}^n	229
a. Convex Combinations and Convex Hull	231
b. Topological Properties of Convex Sets	234
c. Relative Interior and Boundary of a Convex Set	237
d. Separation Theorems	241
2. Concave Functions	245
a. Some Characterizations	246
b. Properties of Concave Functions	251
c. Concavity for Smooth Functions	258
3. Quasiconcave Functions	261
Appendix: Quadratic Forms	268
Bibliography	272
Notes	272
7. Static Optimization	274
1. Nonlinear Programming	274
a. Convex Constraint Set	277
b. Equality Constraints: The Lagrange Problem	282
c. Inequality Constraints: The Kuhn–Tucker Problem	291
d. Concave Programming without Differentiability	297
2. Comparative Statics and Value Functions	300
a. The Theorem of the Maximum	301
b. Comparative Statics of Smooth Optimization Problems	309
c. Value Functions and Envelope Theorems	312
3. Problems and Applications	316
a. Profit Maximization by a Competitive Firm	317
b. Implicit Contracts	319
Bibliography	323
Notes	324

8. Some Applications to Microeconomics	325
1. Consumer Preferences and Utility	327
a. Preference Relations	327
b. Representation by a Utility Function	332
c. Smooth Preferences	338
2. Consumer Theory	339
a. Utility Maximization and Ordinary Demand Functions	340
b. Expenditure Minimization and Compensated Demand	346
c. Relation between Compensated and Ordinary Demands: The Slutsky Equation	352
3. Walrasian General Equilibrium in a Pure Exchange Economy	354
a. Aggregate Demand	356
b. Existence of Competitive Equilibrium	360
c. Welfare Properties of Competitive Equilibrium	368
4. Games in Normal Form and Nash Equilibrium	375
5. Some Useful Models of Imperfect Competition	379
a. Increasing Returns to Specialization in a Dixit–Stiglitz Model	380
b. Fixed Costs, Market Power and Excess Entry in a Cournot Model	383
Bibliography	385
Notes	387

PART III. DYNAMICS

9. Dynamical Systems. I: Basic Concepts and Scalar Systems	391
1. Difference and Differential Equations: Basic Concepts	391
a. Geometrical Interpretation	393
b. Initial- and Boundary-Value Problems	394
c. Some Definitions	396
d. Existence, Uniqueness, and Other Properties of Solutions	398
2. Autonomous Systems	401
a. The Flow of an Autonomous System	402
b. Asymptotic Behavior	408
c. Steady States and Stability	409
3. Autonomous Differential Equations	411
a. Linear Equations with Constant Coefficients	412
b. Nonlinear Autonomous Equations	414
c. A Note on Comparative Dynamics	418
4. Autonomous Difference Equations	419
a. Linear Equations with Constant Coefficients	419
b. Nonlinear Equations	421
5. Solution of Nonautonomous Linear Equations	428
6. Solutions of Continuous-Time Systems	430

a. Local Existence and Uniqueness	431
b. Maximal Solutions	437
c. Dependence on Initial Conditions and Parameters	444
Bibliography	454
Notes	455
10. Dynamical Systems. II: Higher Dimensions	457
1. Some General Results on Linear Systems	457
2. Solution of Linear Systems with Constant Coefficients	459
a. Solution by Diagonalization	460
b. Imaginary Eigenvalues	463
c. Repeated Eigenvalues	465
d. Nonhomogeneous Systems and Stability Conditions	466
e. Stable and Unstable Spaces	470
f. Linear Systems on the Plane	473
3. Autonomous Nonlinear Systems	484
a. Phase Diagrams for Planar Systems	484
b. Local Analysis by Linearization	487
4. Problems	489
Bibliography	491
Notes	492
11. Dynamical Systems III: Some Applications	494
1. A Dynamic IS-LM Model	494
a. Phase Diagram and Stability Analysis	496
b. Effects of Monetary Policy	501
2. An Introduction to Perfect-Foresight Models	503
a. A Model of Stock Prices	503
b. Dornbusch's Overshooting Model	513
3. Neoclassical Growth Models	518
a. Technology and Factor Prices in a Neoclassical World	518
b. The Solow Model	522
c. An Overlapping-Generations Model (Diamond)	527
4. Some Useful Techniques	534
a. Linearization and Derivation of a Convergence Equation	534
b. Solving the Solow Model with <i>Mathematica</i>	538
5. Problems	540
Bibliography	546
Notes	547
12. An Introduction to Dynamic Optimization	549
1. Dynamic Programming	549
a. The Principle of Optimality and Bellman's Equation	550
b. Some Results for Stationary Discounted Problems	558

2. Optimal Control	566
a. The Maximum Principle	567
b. Transversality and Sufficient Conditions	572
c. Constraints Involving State and Control Variables	578
Bibliography	580
Notes	580
13. Some Applications of Dynamic Optimization	582
1. Search Models	582
a. The Basic Model of Job Search	583
b. A Search-Based Macro Model	589
2. Optimal Growth in Discrete Time	598
a. Properties of the Policy Function and the Optimal Capital Sequence	602
b. The Euler Equation and Dynamics	604
3. Investment with Installation Costs	609
a. A Model of Investment with Installation Costs	610
b. Capital Accumulation and Stock Prices in a Small Open Economy	617
4. The Cass–Koopmans Model and Some Applications	622
a. Optimal Consumption for an Infinitely-Lived Household	622
b. Equilibrium and Dynamics in a Model with Taxes on Factor Income	625
c. The Welfare Cost of Factor Taxes	629
5. Problems	643
a. An Efficiency-Wage Model	644
b. Unemployment in a Matching Model	646
c. The Behaviour of the Savings Rate in the Cass–Koopmans Model	647
d. Productive Government Spending in a Model of Endogenous Growth	649
e. A Model of Endogenous R&D	650
Bibliography	653
Notes	654
Appendix. Solutions to the Problems	659
<i>Subject Index</i>	827
<i>Author Index</i>	829

Part I
Preliminaries

1

Review of Basic Concepts

This chapter reviews some basic concepts that will be used throughout the text. One of its central themes is that relations and functions can be used to introduce different types of “structures” in sets. Thus relations of a certain type can be used to order sets according to criteria like precedence, size, or goodness; algebraic operations are defined using functions, and a function that formalizes the notion of distance between two elements of a set can be used to define topological concepts like convergence and continuity. In addition, we also introduce some simple notions of logic and discuss several methods of proof that will be used extensively later on.

1. Sets

A *set* is a collection of objects we call elements. We will denote sets by capital letters, and elements by lowercase letters. If x is an element of a set X , we write $x \in X$, and $x \notin X$ otherwise. A set A is a *subset* of X if all elements of A belong to X . This is written $A \subseteq X$ (A is contained in X). Formally, we can write

$$A \subseteq X \Leftrightarrow (x \in A \Rightarrow x \in X)$$

where the one-way arrow (\Rightarrow) denotes implication, and the two-way arrow (\Leftrightarrow) indicates equivalence. Two sets, A and B , are equal if they have the same elements, that is, $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$. The symbol \emptyset denotes the *empty set*, a set with no elements. By convention, \emptyset is a subset of any set X .

Given a set X , the *power set* of X , written $P(X)$ or 2^X , is the set consisting of all the subsets A of X . A class or *family of sets* in X is a subset of $P(X)$, that is, a set whose elements are subsets of X . We will use “hollow” capital letters to denote families of sets. For example,

$$\mathbb{A} = \{A_i; A_i \subseteq X, i \in I\}$$

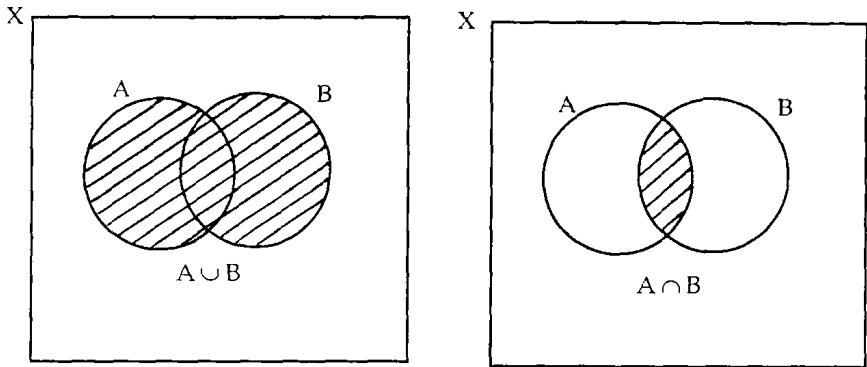


Figure 1.1. Union and intersection of two sets.

where I is some index set, such as the set of all natural numbers smaller than some given number n .

In what follows we will take as given some set X (the universal set), and assuming that “there is nothing” outside X , we will work with its subsets. Given two subsets of X , A and B , we define their *union*, $A \cup B$, as the set

$$A \cup B = \{x \in X; x \in A \text{ or } x \in B\}$$

That is, $A \cup B$ is the set of elements of X that belong to A or to B or to both. Similarly, the *intersection* of A and B , denoted by $A \cap B$, is the set whose elements belong to both sets at the same time:

$$A \cap B = \{x \in X; x \in A \text{ and } x \in B\}$$

These concepts can be extended in a natural way to classes of more than two sets. Given a family of subsets of X , $\mathbb{A} = \{A_i; i \in I\}$, its union and intersection are given respectively by

$$\begin{aligned} \cup \mathbb{A} &= \cup_{i \in I} A_i = \{x \in X; \exists i \in I \text{ s.th. } x \in A_i\} \quad \text{and} \\ \cap \mathbb{A} &= \cap_{i \in I} A_i = \{x \in X; x \in A_i \forall i \in I\} \end{aligned}$$

where the *existential quantifier* “ \exists ” means “there exists some,” the *universal quantifier* “ \forall ” means “for all,” and “s.th.” means “such that.” That is, x is an element of the union $\cup \mathbb{A}$ if it belongs to at least one of the sets in the family \mathbb{A} , and it is an element of the intersection if it belongs to all sets in the class.

The following theorem summarizes the basic properties of unions and intersections of sets.

Theorem 1.1. Properties of unions and intersections of sets. Let A , B , and C be three subsets of X . Then the following properties hold: