COMBINATORIAL MATHEMATICS

H. J. RYSER

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COMBINATORIAL MATHEMATICS

By

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PREFACE

This monograph requires no prior knowledge of combinatorial mathematics. In Chapter 1 we deal with the elementary properties of sets and define *permutation*, *combination*, and *binomial coefficient*. Of course we treat these concepts from a mature point of view, and from the outset we assume an appreciation for the subtleties of mathematical reasoning. Combinatorial mathematics is best studied within the framework of modern algebra, and for this reason we presuppose a certain familiarity with a few algebraic concepts. Matrices are the really important tool. They occur throughout the monograph and unify the various chapters. At first they are primarily rectangular arrays and little is needed in the way of background. Later they play a fuller role, and we apply the standard rules of matric manipulation. Number theory is used sparingly. An understanding of integral congruences is adequate for most purposes. Groups and fields are mentioned in passing. Only on rare occasions do we call for something beyond the definitions of these systems.

Many of our proofs rely on counting arguments, finite induction, or some other time-tested device. But this does not mean that combinatorial mathematics is easy. The subject is demanding and its exposition is troublesome. Our definitions and proofs are concise and they deserve careful scrutiny. But effort and ingenuity lead to mastery, and our subject holds rich rewards for those who learn its secrets.

We pursue certain topics with thoroughness and reach the frontiers of present-day research. But we pay a price for this and must omit much that is of interest. Each chapter contains its separate bibliography. These are guides for further study and do not aim at completeness. We also discuss in the pages that follow some vital questions that remain unanswered. Combinatorial mathematics is tremendously alive at this moment, and we believe that its greatest truths are still to be revealed.

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CONTENTS

CHAPTER 1

FUNDAMENTALS OF COMBINATORIAL MATHEMATICS

1. What is combinatorial mathematics? Combinatorial mathematics, also referred to as *combinatorial analysis* or *combinatorics*, is a mathematical discipline that began in ancient times. According to legend the Chinese Emperor Yu (c. 2200 B.C.) observed the magic square

$$\begin{bmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{bmatrix}$$

on the back of a divine tortoise. Permutations had a feeble beginning in China before 1100 B.C., and Rabbi Ben Ezra (c. 1140 A.D.) seems to have known the formula for the number of combinations of n things taken r at a time. Much of the earliest work is tied to number mysticism. But during the last few centuries various writers have approached the subject from the standpoint of mathematical recreations. Bachet's problem of the weights, Kirkman's schoolgirls problem, and Euler's 36 officers problem are famous illustrations. Such problems are intellectually stimulating, and their solutions are sometimes ingenious and elegant.

Many of the problems studied in the past for their amusement or aesthetic appeal are of great value today in pure and applied science. Not long ago finite projective planes were regarded as a combinatorial curiosity. Today they are basic in the foundations of geometry and in the analysis and design of experiments. Our new technology with its vital concern with the discrete has given the recreational mathematics of the past a new seriousness of purpose.

But more important, the modern era has uncovered for combinatorics a wide range of fascinating new problems. These have arisen in abstract algebra, topology, the foundations of mathematics, graph theory, game theory, linear programming, and in many other areas. Combinatorics has always been diversified. During our day this diversification has increased manyfold. Nor are its many and varied problems successfully attacked in terms of a unified theory. Much of what we have said up to now applies with equal force to the theory of numbers. In fact, combinatorics and number theory are sister disciplines. They share a certain intersection of common knowledge, and each genuinely enriches the other.

Combinatorial mathematics cuts across the many subdivisions of mathematics, and this makes a formal definition difficult. But by and large it is concerned with the study of the arrangement of elements into sets. The elements are usually finite in number, and the arrangement is restricted by certain boundary conditions imposed by the particular problem under investigation. Two general types of problems appear throughout the literature. In the first the existence of the prescribed configuration is in doubt, and the study attempts to settle this issue. These we call existence problems. In the second the existence of the configuration is known, and the study attempts to determine the number of configurations or the classification of these configurations according to types. These we call enumeration problems. This monograph stresses existence problems, but many enumeration problems appear from time to time.

It may be remarked that the second category of problems is nothing more than a refinement or obvious extension of the first. This is the case. But in practice if the existence of a configuration requires intensive study, then almost nothing can be said about the corresponding enumeration problem. On the other hand, if the enumeration problem is tractable, the corresponding existence problem is usually trivial.

We illustrate these remarks with an elementary example. An 8 by 8 checkerboard has 2 squares from opposite corners removed. There are available 31 dominoes, and each domino covers exactly 2 checkerboard squares. The problem is to cover the entire board with the 31 dominoes. In this problem the existence of a solution is in doubt. We show in fact that such a covering is not possible. For 2 black or 2 white squares are deleted. Thus the board has an unequal number of black and white squares. But a domino placed on the board must cover both a black and a white square. Therefore a complete covering is not possible. Suppose the 2 squares from opposite corners are not deleted. Then it is possible to cover the board with 32 dominoes in many ways. Under these circumstances one is led to the enumeration problem of determining the number of distinct coverings.

2. Sets. Let S be an arbitrary set of elements a, b, c, ... We indicate the fact that s is an element of S by writing $s \in S$. If each element of a set A is an element of the set S, then A is a *subset* of S and we designate this by the notation $A \subseteq S$. If $A \subseteq S$ and $S \subseteq A$, then the two sets are identical and we write $A = S$. If $A \subseteq S$ but $A \neq S$, then A is a *proper subset* of S and we write $A \subset S$. The set of all subsets of S is denoted by $P(S)$. For notational convenience the vacuous set or *null set* $\emptyset$ is counted as a member of $P(S)$.

Let S and T be subsets of a set M. The set of elements e such that $e \in S$ and $e \in T$ is called the *intersection* $S \cap T$ of S and T. More generally, if $T_1, T_2, \ldots, T_r$ are subsets of M, then $T_1 \cap T_2 \cap \cdots \cap T_r$ denotes the set of elements e such that $e \in T_i$ for each $i = 1, 2, \ldots, r$. The subsets S and T of M are *disjoint* provided they have no elements in common. The equation $S \cap T = \emptyset$ indicates that S and T are disjoint. The *union* $S \cup T$ of the subsets S and T of M is the set of elements e such that $e \in S$ or $e \in T$. More generally, if $T_1, T_2, \ldots, T_r$ are subsets of M, then $T_1 \cup T_2 \cup \cdots \cup T_r$ denotes the set of all elements e such that $e \in T_i$ for at least one $i = 1, 2, \ldots, r$. The subsets $T_1, T_2, \ldots, T_r$ of M form a *partition* of M provided $M = T_1 \cup T_2 \cup \cdots \cup T_r$ and $T_i \cap T_j = \emptyset$ for $i \neq j$ $(i, j = 1,2, \ldots, r)$. The partitions of M are *ordered* if equality of the partitions $M = T_1 \cup T_2 \cup \cdots \cup T_r$ and $M = T'_1 \cup T'_2 \cup \cdots \cup T'_r$ means that $T_i = T'_i$ $(i = 1, 2, \ldots, r)$ and *unordered* if equality of the partitions means that each T_i is equal to some T'_j.

A set S containing only a finite number of elements is called a *finite set*. A finite set is a *set of n elements* provided the number of its elements is n. When this terminology is used we take $n > 0$ and exclude the null set $\emptyset$. *Throughout the monograph we call a set of n elements an n-set.* Thus an r-subset of an n-set means a subset of r elements of a set of n elements. Many counting arguments make extensive use of the following elementary principles.

Let S be an m-set and let T be an n-set. If $S \cap T = \emptyset$, then $S \cup T$ is an $(m + n)$-set. This is the *rule of sum.* The *generalized rule of sum* asserts the following. *If T_i is an n_i-set $(i = 1, 2, \ldots, r)$ and if $M = T_1 \cup T_2 \cup \cdots \cup$*

T_r is a partition of M, then M is an $(n_1 + n_2 + \cdots + n_r)$-set.

Let S and T denote two sets and let (s, t) be an ordered pair with $s \in S$ and $t \in T$. Two pairs (s, t) and (s^*, t^*) are equal if $s = s^*$ and $t = t^*$. The set of all of these ordered pairs is called the *product set* of S and T and is denoted by $S \times T$. Let $M(S, T, n)$ denote a set of ordered pairs of the form (s, t), where s is arbitrary in S but each $s \in S$ is paired with exactly n elements $t \in T$. Distinct elements of S need not be paired with elements of the same n-subset of T. The notation implies T contains at least n elements. Moreover, $M(S, T, n) = S \times T$ if and only if T is an n-set. Now let S be an m-set. Then $M(S, T, n)$ is an (mn)-set. This is the *rule of product*. The *generalized rule of product* asserts the following. *If T_1 is an n_1-set and if $M_2 = M(T_1, T_2, n_2)$, $M_3 = M(M_2, T_3, n_3)$, and finally if $M_r = M(M_{r-1}, T_r, n_r)$, then M_r is an $(n_1 n_2 \ldots n_r)$-set.*

3. Samples. Let S be a set and let

$$(a_1, a_2, \ldots, a_r) \tag{3.1}$$

be an ordered r-tuple of not necessarily distinct elements of S. Two such r-tuples $(a_1, a_2, \ldots, a_r)$ and $(a_1^*, a_2^*, \ldots, a_r^*)$ are equal if $a_i = a_i^*$ $(i = 1, 2, \ldots, r)$. We call (3.1) a *sample* of S. The sample is of *size* r, and we refer to (3.1) as an *r-sample* of S.

THEOREM 3.1. *The number of r-samples of an n-set is n^r.*

Proof. Let S be an n-set. This theorem is a special case of the generalized rule of product with $T_1 = T_2 = \cdots = T_r = S$ and $n_1 = n_2 = \cdots = n_r = n$.

Let S be an n-set and let the components a_i of the r-sample (3.1) be distinct. Then the r-sample is called an *r-permutation* of n elements. An r-permutation must have $r \leq n$. An n-permutation is called a *permutation* of n elements.

THEOREM 3.2. *The number of r-permutations of n elements is*

$$P(n, r) = n(n-1)\cdots(n-r+1). \tag{3.2}$$

Proof. This theorem is a special case of the generalized rule of product with $T_1 = T_2 = \cdots = T_r = S$ and $n_1 = n$, $n_2 = n-1, \ldots, n_r = n-r+1$.

By (3.2) $P(n, n)$ stands for the product of the first n positive integers. $P(n, n)$ is called *n-factorial* and is written $n!$. Thus

$$P(n, n) = n! = n(n-1)\cdots 1. \tag{3.3}$$

COROLLARY 3.3. *The number of permutations of n elements is n!.*

A (single-valued) *mapping* α of a set S *into* a set T is a correspondence that associates with each $s \in S$ a unique $t = s\alpha \in T$. The element $s\alpha$ is called the *image* of s under the mapping α. Two mappings α and β of S into T are equal if $s\alpha = s\beta$ for all $s \in S$. The mapping α is a mapping of S *onto* T if every $t \in T$ occurs as an image of some $s \in S$. The mapping of S onto T is 1-1 if distinct elements of S have distinct images. Now let $G(S)$ be the set of all 1-1 mappings of S onto itself. Let α and β be in $G(S)$. Then the mapping that sends $s \in S$ into $(s\alpha)\beta \in S$ is a 1-1 mapping called the *product* of the mappings α and β. $G(S)$ is now an algebraic system with a binary composition called product, and one may verify that $G(S)$ satisfies the axioms of a group.

Let S be an n-set of elements labeled $1, 2, \ldots, n$. Then $G(S)$ is called the *symmetric group* of *degree n*. It is denoted by S_n. Let α be the element in S_n that sends i into $i\alpha$ $(i = 1, 2, \ldots, n)$. The 1-1 mapping α is characterized by the permutation

$$(1\alpha, 2\alpha, \ldots, n\alpha). \tag{3.4}$$

Conversely, each permutation of the n elements is in effect a 1-1 mapping of the elements onto themselves. The number of elements in a group is called its *order*. We may now state Corollary 3.3 in the terminology of group theory.

COROLLARY 3.4. *S_n is of order $n!$.*

Examples. (a) The number of 2-permutations of 3 elements is $P(3, 2) = 3 \cdot 2 = 6$. If the elements are labeled 1, 2, 3, the 2-permutations are

$$(1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2).$$

(b) The number of 5-letter words that may be constructed out of the English alphabet is 26^5. The number of 5-letter words on distinct letters is

$$26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 = 7{,}893{,}600.$$

(c) S_{100} is of order $(9.3326 \ldots)10^{157}$. Eddington's estimate of the number of electrons in the universe is a mere $(136)2^{256}$.

(d) Let A be a matrix of m rows and n columns, and let the entries of A be the integers 0 and 1. There are 2^{mn} of these matrices. If $m = n = 100$, this gives $2^{10,000}$ matrices.

4. Unordered selections.

Let S be a set and let

$$\{a_1, a_2, \ldots, a_r\} \tag{4.1}$$

be an unordered collection of r not necessarily distinct elements of S. The number of occurrences of an element in the collection is called the *multiplicity* of the element. Two such collections $\{a_1, a_2, \ldots, a_r\}$ and $\{a_1^*, a_2^*, \ldots, a_r^*\}$ are equal provided the elements with their respective multiplicities are the same for both collections. We call (4.1) an *unordered selection* of S. The unordered selection is of *size* r, and we refer to (4.1) as an *r-selection* of S. If each element in (4.1) is of multiplicity 1, then the r-selection is an r-subset of S. An r-subset of an n-set is also called an *r-combination* of n elements.