Pseudodifferential Operators

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Pseudodifferential Operators

Introduction

This book develops a circle of techniques used to treat linear partial differential equations

(I.1) $$Pu = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha u = f$$

on a region Ω, generally supplemented by boundary conditions on one or more hypersurfaces in Ω. The three main classical examples of (I.1) are the following:

(I.2) The Laplace equation $\Delta u = f$

where

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \quad \text{on} \quad \Omega \subset \mathbf{R}^n.$$

This is typically supplemented by the Dirichlet boundary condition $u|_{\partial\Omega} = g$ or the Neumann boundary condition $(\partial u/\partial \nu)|_{\partial\Omega} = g$, though other boundary conditions also occur.

(I.3) The heat equation $\dfrac{\partial}{\partial t} u = \Delta u, \quad t \in \mathbf{R}^+, \quad x \in \Omega.$

This is typically supplemented by an initial condition $u(0, x) = f(x)$ and, if Ω has nonempty boundary, a boundary condition on $\mathbf{R}^+ \times \partial\Omega$ such as a Dirichlet or Neumann boundary condition.

(I.4) The wave equation $\left(\dfrac{\partial^2}{\partial t^2} - \Delta\right) u = 0, \quad t \in \mathbf{R}, \quad x \in \Omega.$

This is typically supplemented by the initial condition $u(0, x) = f_1(x)$, $u_t(0, x) = f_2(x)$, and if $\partial\Omega \neq \varnothing$ a boundary condition as in the first two cases.

Equations (I.2)–(I.4) are said to be of elliptic, parabolic, and hyperbolic type, respectively. These equations and natural generalizations make up a large portion of the linear partial differential equations of mathematical physics, and their theory suggests questions to be asked about general linear *PDE*. These questions are typically in one of three categories: existence, uniqueness, qualitative behavior.

For the purposes of mathematical physics, the first two questions may be considered preliminary (though not necessarily trivial from a mathematical viewpoint); after all, the point of the application of calculus to physics is not to prove that real processes actually occur but rather to describe the nature of such occurrences. Thus the third category forms the core of classical linear *PDE*. Within this category, many questions arise; one asks what the solutions look like, and ideally one would want to know everything about them. Such properties as regularity, location of singularities, and estimates in various norms are important examples, particularly emphasized in this treatment, and more questions arise such as on the spectral behavior of P, decay of solutions, location of maxima or nodal sets, limiting behavior under (possibly quite singular) perturbations of the equation or the boundary, and many more. Numerous such questions are addressed in these pages. We also derive existence and uniqueness theorems adequate for many linear equations of mathematical physics, and most such results are fairly simple (unfortunately, one cannot say the same for nonlinear *PDE* of mathematical physics).

To put in perspective the role of pseudodifferential operators in the study of linear *PDE*, we list four tools of linear *PDE*.

(1) *Functional analysis.* The use of various Hilbert, Banach, Frechet, and LF spaces is all pervasive in modern linear *PDE*. Sobolev spaces and spaces of distributions, described in Chapter I, form the setting for most of the analysis, though other spaces, particularly Hölder spaces and Besov spaces (discussed in Chapter XI) make an occasional appearance.

(2) *Fourier analysis.* The use of Fourier series and/or the Fourier transform in constant coefficient PDE is intimately connected with separation of variables, and Fourier analysis was used by Daniel Bernoulli in the very beginning of the study of the subject. In the modern approach, Fourier analysis, via the Plancherel theorem particularly, is often used to get estimates of solutions to constant coefficient equations, obtained from variable coefficient equations by freezing the coefficients and, if a boundary is present, flattening out the boundary. One then patches these estimates together to get estimates in the variable coefficient situation.

(3) *Energy estimates.* This includes integral estimates on quadratic forms of a function and its derivatives. One of the basic estimates of this sort is Gårding's inequality

$$Re(Pu, u) \geq c_1\|u\|_{H^m}^2 - c_2\|u\|_{L^2}^2, \qquad u \in C_0^\infty(\Omega) \tag{I.5}$$

for a partial differential operator

$$P = \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha,$$

assuming

$$Re\, P_{2m}(x, \xi) = Re\left(\sum_{|\alpha| = 2m} a_\alpha(x)\xi^\alpha \right) \geq c|\xi|^{2m}.$$

Here

$$\|u\|_{H^m}^2 = \sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^2}^2$$

defines a norm on a space H^m known as a Sobolev space (discussed in Chapter I). When P is a second order scalar differential operator, (I.5) is proved simply by integration by parts. For higher order operators, one way to get (I.5) is to freeze coefficients, obtain such an estimate for constant coefficient operators by Fourier analysis, and glue these estimates together. Gårding's inequality for pseudodifferential operators, generalizing (I.5), will be proved in Chapter II, Section 8, using the calculus of pseudodifferential operators. Energy estimates are also used to prove existence, uniqueness, and finite propagation speed for solutions to hyperbolic equations, and it is in the study of second order hyperbolic equations that the term *energy* has a most direct physical interpretation. Gårding's inequality plays a great role in unifying many of these energy estimates, as will be seen in Chapters IV and V. Weighted L^2 estimates, known as Carleman estimates, have also played an important role in linear *PDE*, though they are not much emphasized here, except in Chapter XIV which considers uniqueness in the Cauchy problem.

(4) *Fundamental solutions and parametrices.* A distribution $E(x, y)$ such that

$$P(x, D_x)E = \delta(x - y) \tag{I.6}$$

is called a fundamental solution. If the two sides of (I.6) differ by a smooth function, E is called a parametrix. One can get a lot of information about solutions to (I.1) if one of these is known. For variable coefficient equations it is usually impossible to construct a fundamental solution; one is generally happy to be able to construct a parametrix. One way to do this for elliptic operators is the following. A first order approximation to a parametrix is obtained by freezing coefficients and dropping lower order terms, getting fundamental solutions for such simple equations, via Fourier analysis, and gluing these together. A true parametrix is then obtained by an iterative procedure. This technique is often called the Levi parametrix method, and one of the simplest applications of the calculus of pseudodifferential operators, as we shall see in Chapter III, is to carry out such a procedure. For nonelliptic operators, much more subtle methods are required to construct parametrices. Developing such methods is one of the main themes of this book.

We begin in Chapter I with a summary of the basic facts about distributions and Sobolev spaces most frequently used in *PDE*. There are numerous excellent books giving more leisurely and complete treatments of this subject, and we mention Yosida's *Functional Analysis* [1], Gelfand et al. *Generalized Functions* [1], Donoghue's *Distribution Theory* [1], Adam's *Sobolev Spaces* [1], and particularly the first part of Lions and Magene's work on boundary value problems [1]. The reader is assumed to be familiar with functional analysis and should have some understanding of distribution theory.

Our subject proper starts with Chapter II, where pseudodifferential operators are defined and some of their basic properties are studied, such as the behavior of products and adjoints of such operators, their continuity on L^2 and Sobolev spaces, the fact that they do not increase the singular support of distributions to which they are applied, and the Gårding inequality, generalizing (I.5). In Chapters III through V this calculus of pseudodifferential operators is applied to some basic questions of existence and regularity of solutions to elliptic, hyperbolic, and parabolic equations, and elliptic boundary value problems. The regularity theorem for elliptic differential operators P of order m, defined as those for which $|P_m(x, \xi)| \geq c|\xi|^m$, states that if Pu is C^∞ on an open set U, $u \in C^\infty(U)$, and more generally if Pu possesses a certain degree of smoothness, in an appropriate space, then u possesses m more degrees of smoothness. An operator P, for which u is C^∞ wherever Pu is with perhaps u possessing fewer than m extra orders of smoothness then Pu generally, is called hypoelliptic, and some results on hypoelliptic operators are given in Chapter III, though further, deeper results are given in Chapter XV. In Chapter IV one application is given to a topic in nonlinear *PDE*: the short time existence of solutions to quasi-linear hyperbolic equations.

In Chapter VI a new concept, that of the wave front set of a distribution, is introduced. The wave front set of a distribution on U is a subset of the cotangent bundle T^*U, lying over the singular support of u, and thus is a refinement of the notion of singular support. It turns out to be the natural language for stating theorems on the propagation of singularities of solutions to PDE, and Chapter VI gives a proof of Hörmander's theorem on propagation of singularities. This proof requires a new tool, the sharp Gårding inequality, proved in Chapter VII.

In Chapter VIII a new theme is taken up, the calculus of Fourier integral operators, a class more general than pseudodifferential operators. Such operators are useful for constructing parametrices for many operators that are not hypoelliptic, in particular hyperbolic operators, and their use extends classical methods of geometrical optics. This study makes use of some basic notions of the symplectic form and Hamiltonian vector fields. Some excellent references for this aspect of advanced calculus include Abraham and Marsden's *Foundations of Mechanics* [1] and Arnold's *Mathematical Methods of Classical Physics* [1], and also Caratheodory's classic [1]. The Hamiltonian vector field associated with the principal symbol of a pseudodifferential operator (of classical type) generates a flow on the cotangent bundle called the bicharacteristic flow, and Hörmander's propagation of singularities theorem asserts that if $Pu = f \in C^\infty(U)$, then the wave front set of u is invariant under this bicharacteristic flow. A second proof of this result is given in Chapter VIII, and in Chapters IX and X propagation of singularities for solutions to boundary value problems is discussed, first in the case of bicharacteristics transversal to the boundary and then for bicharacteristics that "graze" the boundary, being convex with respect to the boundary.

In Chapter XI we study the behavior of various classes of pseudodifferential operators on L^p and Hölder spaces and include a treatment of estimates for solutions to regular elliptic boundary value problems (discussed in Chapter V in the L^2 context) within these categories. In this chapter we make use of results of Marcinkiewicz, Mikhlin, and Hörmander on continuity of certain Fourier multipliers on $L^p(\mathbf{R}^n)$. Stein's book *Singular Integrals and Differentiability Properties of Functions* [1] provides a thorough treatment of this material.

Chapter XII studies questions about the eigenvalues and eigenfunctions of an elliptic self adjoint operator on a compact manifold, including eigenvalue asymptotics and convergence of eigenfunction expansions, and some applications to harmonic analysis on compact Lie groups, among other things. Fourier integral operators provide the tool for studying functions of an elliptic self-adjoint operator. This study in turn is the key to a systematic generalization of many topics in classical harmonic analysis on the torus. Several alternative approaches are given to some results, some of which involve using Tauberian theorems, and we include an appendix on Wiener's Tauberian theorem and some consequences. Generally, I have preferred to refer to other books where certain prerequisite material has been well treated, to keep this work from becoming enormous. However, it seems to me that most expositions of Wiener's Tauberian theorem encourage the reader to avoid perceiving the nature of Wiener's inspiration—namely that his closure of translates theorem was a very simple result that wonderfully tied together many seemingly diverse Tauberian theorems, of which the Hardy-Littlewood and Karamata theorems are important and most frequently used examples.

Chapter XIII is devoted to the Calderon-Vaillancourt theorem on L^2 boundedness of pseudodifferential operators in a borderline case not covered in Chapter II, and to Hörmander-Melin inequalities, on the semiboundedness of second order pseudodifferential operators. Chapter XIV gives some results on uniqueness in the Cauchy problem: given a hypersurface Σ dividing U into two parts, U^+ and U^-, when does $Pu = 0$ on U, $u = 0$ on U^+ imply $u = 0$ on a neighborhood of Σ?

These first fourteen chapters deal primarily with operators which are either elliptic or whose characteristics are simple. The final chapter studies operators with double characteristics. Included here are such hypoelliptic operators as arise in the analysis of the $\bar{\partial}$-Neumann problem in strictly pseudoconvex domains and also certain equations of mathematical physics, such as the equations of crystal optics.

Chapters II through VII have been covered by the author in a one semester course at the University of Michigan, for students who the previous semester had covered some basic *PDE*, including the Sobolev space theory and Fourier analysis given in Chapter I. A preliminary version of these chapters was published as a Springer Lecture Note (Taylor [3]). Most of the material in Chapters VIII through XV has been covered by the author in various courses and seminars at Michigan, Stony Brook, the Courant Institute, and Rice University. The entire book contains more material than one would cover in a year's course, and it should be noted that not every chapter depends on all the previous ones. Chapter XI on L^p and Hölder estimates could be read directly after Chapter II. If one wanted to get quickly to Chapter VIII on Fourier integral operators, one could skip Chapter V on elliptic boundary value problems and the proof of propagation of singularities in Chapter VI using the sharp Gårding inequality, and hence also Chapter VII. As for Chapter XIII, the Claderon-Vaillancourt theorem could be treated directly after Chapter II, and the Hörmander-Melin inequality after Chapter VIII. Chapter XIV and almost all of Chapter XV could be read right after Chapter VIII, the Hörmander-Melin inequality playing a role in some results of the first section of Chapter XV.

The treatment of Fourier integral operators emphasizes the local theory. I avoid discussing an invariantly defined symbol of a Fourier integral operator and in particular do not introduce the Keller-Maslov line bundle. For a complementary treatment of Fourier integral operators which does emphasize such global techniques, I recommend Duistermaat's *Fourier Integral Operators* [1]. Also the reader should consult the original paper of Hörmander [14], of Duistermaat and Hörmander [1], and the book of Guillemin and Sternberg [1].

The beginning student of partial differential equations should be aware that many other methods have been brought to bear besides those connected with pseudodifferential operators. This present book is not intended as a general introduction to the subject of partial differential equations, and we want to give the reader some guide to a set of books which, together, would provide a fairly complete introduction. First, some general texts introducing most of the basic problems of the classical theory, on a fairly elementary level, are: R. Courant and D. Hilbert, *Methods of Mathematical Physics*, vol. 2 [1]; and P. Garabedian, *Partial Differential Equations* [1]. More complete references are given in the bibliography.

Some books that cover a more restricted class of problems, using more advanced techniques, particularly functional analysis and energy estimates, include: L. Bers, F. John and M. Schechter, *Partial Differential Equations*, [1]; L. Hörmander, *Linear Partial Differential Operators*, [16]; S. Mizohata, *The Theory of Partial Differential Equations* [1]; and, F. Trèves, *Linear Partial Differential Equations with Constant Coefficients* [1]. For a treatment of elliptic operators, see: S. Agmon, *Lectures on Elliptic Boundary Value Problems* [2]; D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of the Second Order* [1]; J. Lions and E. Magenes, *Inhomogeneous Boundary Value Problems and Applications* [1]; C. B. Morrey, *Multiple Integrals in the Calculus of Variations* [1]; and M. Protter and H. Weinberger, *Maximum Principles in Differential Equations* [1]. The books by Gilbarg and Trudinger and by Morrey give thorough treatments of quasilinear elliptic equations.

For the connection between the heat equation and diffusion processes, see: A. Friedman, *Stochastic Differential Equations and Applications* [2]; and H. McKean, *Stochastic Integrals* [1]. For the study of scattering theory, quantum mechanical and classical, respectively, I refer to: W. Amrein, J. Jauch, and K. Sinha, *Scattering Theory in Quantum Mechanics* [1]; and P. Lax and R. Phillips, *Scattering Theory* [1]. Some results on nonlinear wave equations are given in R. Courant and K. Friedrichs, *Supersonic Flow and Shock Waves* [61]; and G. Whitham, *Linear and Nonlinear Waves* [1]. Finally, one should see an old fashioned treatment of *PDE* using mainly Green's formula and separation of variables. A good one is I. Stakgold, *Boundary Value Problems of Mathematical Physics* [1]; and good treatments of the special function theory created to treat such problems are given in N. Levedev, *Special Functions and their Applications* [1]; and F. Olver, *Asymptotics and Special Functions* [1].

It seems quite likely that special function theory will make a comeback in *PDE* as modern breakthroughs allow one to use such functions in a more sophisticated way than was done during the last century, even as today more sophisticated use is made of the exponential function than in the eighteenth century. The only higher transcendental function used in this book is the Airy function, which appears in Chapter X.

CHAPTER I

Distributions and Sobolev Spaces

Distributions and Sobolev spaces form a most convenient vector by which methods of functional analysis are brought to bear on problems in partial differential equations, and Fourier analysis plays an enormous role. The basic results used in most of this book are summarized in Sections 1–5. Brief proofs of most of the crucial results are given and references are provided for further results. For more leisurely treatments of Sobolev spaces the reader might refer to Trèves [1], Yosida [1], Bergh and Löfström [1], Lions and Magenes [1], or Adams [1]. Of course, Schwartz [1] and Gelfand et al. [1] are the classics for distribution theory; see also Donoghue [1]. The treatment here owes a great deal to the expositions of Yosida and Hörmander [6], except that I emphasize the interpolation method of A. Calderon. The sixth section treats very briefly Sobolev spaces associated with L^p. Such spaces occur only in Chapters XI and XII of this book. I refer particularly to Stein [1] for analysis on L^p.

I agree with Sternberg [1] that any long list of concepts should be accompanied by some nontrivial theorem, so in Section 7 some basic Fourier analysis is applied to prove local solvability of partial differential equations with constant coefficients.

§1. Distributions

Here we define certain spaces of smooth functions, $C^\infty(\Omega)$ and $C_0^\infty(\Omega)$, and their dual spaces, $\mathscr{E}'(\Omega)$ and $\mathscr{D}'(\Omega)$, which are spaces of distributions. We let Ω be an open subset of $\mathbf{R}^n$, or more generally a smooth paracompact manifold. Suppose $\Omega = \bigcup \Omega_j$ where each Ω_j is open and has compact closure in Ω_{j+1}. When $\Omega = \mathbf{R}^n$, we shall define the Schwartz space $\mathscr{S}$ of rapidly decreasing functions, and its dual $\mathscr{S}'$, the space of tempered distributions.

For a compact subset K of Ω and a nonnegative integer j, define a seminorm $p_{K,j}$ on smooth functions by

$$p_{K,j}(u) = \sup_{x \in K} \{|D^\alpha u(x)| : |\alpha| \le j\} \tag{1.1}$$