Classical Electromagnetic Theory

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Preface

This book is based on the author's lecture notes for a graduate course in electricity and magnetism which he gave at Hofstra College in the years 1954 to 1961. It is intended to serve as a textbook explaining the physical concepts of electricity and magnetism, describing the mathematical formalism, and presenting examples of both the ideas and methods involved.

An effort was made to keep the book self-contained. For this reason much of the material usually covered in an advanced undergraduate course in electrostatics has been included. Chapter 1 deals with the fundamentals of vector and tensor analysis. The reader is assumed to have no more mathematical background than that provided by undergraduate courses in advanced calculus and ordinary differential equations. More sophisticated mathematical methods are developed in the text as required. In this way it is hoped that the text will also prove suitable for both the ambitious undergraduate senior and the inadequately prepared graduate student.

As stated above, Chap. 1 treats the elements of vector and tensor analysis. The next five chapters deal with the fundamental ideas of electrostatics. Chapter 7 treats the special theory of relativity. Its main purpose is to lay the foundation for an introduction to the concept of the magnetic field in such a manner as to stress its origin in the motion of charges. The properties of the magnetic field are treated in Chap. 8. Chapter 9 deals with the derivation of Maxwell's equations and the wave equations. Chapters 10 to 13 are concerned with the propagation of plane, spherical, and cylindrical electromagnetic waves. Cavity resonators and wave guides are treated in Chaps. 14 and 15, respectively. Chapter 16 deals with the Lagrangian and Hamiltonian formulation of the electromagnetic field. Electron theory is treated in Chap. 17.

Rationalized mks units are used throughout the text. Tables for the conversion of units and equations to other unit systems are given in Appendixes IV and V.

Problems are listed at the end of each chapter. They are sometimes used to treat material omitted from the text because of space limitations. References are also listed at the end of each chapter.

The author wishes to express his gratitude to Mrs. Claudia Quickel, who typed the manuscript. He is indebted to his former students and his colleagues for encouragement and suggestions. He is especially indebted to Professor Lawrence A. Wills of the City College of New York for his careful perusal and criticism of the manuscript.

The author welcomes all suggestions for the improvement of the text.

Nunzio Tralli

Contents

CHAPTER 1

Scalars, Vectors, and Tensors

1-1. INTRODUCTION

In this chapter are presented those elements of vector and tensor analysis which will be found useful in the development of the subject matter in the remaining chapters of the text. No attempt at mathematical rigor has been made. It is hoped that the presentation will serve as a refreshing review for those who have already studied vector and tensor analysis and as a pleasant introduction for the novice.

1-2. VECTORS

A *vector* is a physical quantity which possesses both a magnitude and a direction. Such a quantity can be described mathematically by means of a *representative* in a cartesian coordinate system. In each cartesian coordinate system there is a unique representative of the vector.

To be specific, let **A** denote a vector. In the $OXYZ$ cartesian coordinate system its representative is the line segment $\vec{A}$ with components A_x, A_y, A_z. In another coordinate system, say $O'X'Y'Z'$, the representative of **A** is $\vec{A}'$ with components A'_x, A'_y, A'_z. The components of $\vec{A}$ and $\vec{A}'$ are related by

$$\begin{aligned} A'_x &= a_{xx}A_x + a_{xy}A_y + a_{xz}A_z \\ A'_y &= a_{yx}A_x + a_{yy}A_y + a_{yz}A_z \\ A'_z &= a_{zx}A_x + a_{zy}A_y + a_{zz}A_z \end{aligned} \tag{1-1}$$

and

$$\begin{aligned} A_x &= a_{xx}A'_x + a_{yx}A'_y + a_{zx}A'_z \\ A_y &= a_{xy}A'_x + a_{yy}A'_y + a_{zy}A'_z \\ A_z &= a_{xz}A'_x + a_{yz}A'_y + a_{zz}A'_z \end{aligned} \tag{1-2}$$

where the a's are direction cosines. For example, a_{xy} is the cosine of the angle between $O'X'$ and OY.

The transformation laws (1-1) and (1-2) define a vector quantity. That is, if the representative of a physical quantity in any cartesian coordinate system has three components and these components transform according to the laws (1-1) and (1-2), the physical quantity is a vector.

Before proceeding further, it is convenient to introduce a more concise notation. Denote the OX axis by OX_1, the OY axis by OX_2, and the OZ axis by OX_3 with a like notation in the primed coordinate system. Then a_{xx} is denoted by a_{11}, a_{xy} by a_{12}, a_{xz} by a_{13}, etc., and Eqs. (1-1) and (1-2) can be written

$$A_i' = \sum_{j=1}^{3} a_{ij}A_j \qquad i = 1, 2, 3 \tag{1-1a}$$

$$A_i = \sum_{j=1}^{3} a_{ji}A_j' \qquad i = 1, 2, 3 \tag{1-2a}$$

The notation can be further simplified to

$$A_i' = a_{ij}A_j \tag{1-1b}$$

$$A_i = a_{ji}A_j' \tag{1-2b}$$

where it is to be understood that, whenever a literal suffix or index appears twice in a term, that term is to be summed for values of the suffix 1, 2, 3. Hence, since j appears twice, the summation $\sum_{j=1}^{3}$ is indicated. Furthermore, since a repeated index, often called a dummy index, indicates summation, another letter can be substituted for it at will.

Substitution for A_j in (1-1b) by means of (1-2b) yields

$$A_i' = a_{ij}a_{kj}A_k'$$

while substitution for A_j' in (1-2b) by means of (1-1b) yields

$$A_i = a_{ji}a_{jk}A_k$$

Hence the direction cosines satisfy the relations

$$a_{ij}a_{kj} = \delta_{ik} \tag{1-3}$$

and

$$a_{ji}a_{jk} = \delta_{ik} \tag{1-4}$$

where δ_{ik} is known as the *Kronecker delta* and has the values

$$\delta_{ik} = 1 \quad \text{when } i = k$$
$$= 0 \quad \text{when } i \neq k$$

Another much-used notation is the matrix notation. In this notation Eq. (1-1a) or (1-1b) is written

$$\begin{pmatrix} A_1' \\ A_2' \\ A_3' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \tag{1-1c}$$

The 3 × 3 array

$$\underline{\underline{a}} \equiv \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \tag{1-5}$$

is known as a 3 × 3 *matrix* or as a square matrix of order 3^2. The quantities a_{ij} are called the *elements* of the matrix. The quantity

$$\underline{A} \equiv \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \tag{1-6}$$

is called a 3-*column* or a 3 × 1 matrix. Its elements A_i are the components of $\vec{A}$, the representative of the vector **A** in the $OX_1X_2X_3$ coordinate system. The multiplication of a 3-column by a 3 × 3 matrix from the left is defined by Eqs. (1-1*a*) and (1-1*c*).

In this matrix notation Eqs. (1-1*a*) and (1-2*a*) can be written

$$\underline{A}' = \underline{\underline{a}}\underline{A} \tag{1-1d}$$

$$\underline{A} = \underline{\underline{a}}^T\underline{A}' \tag{1-2c}$$

where the matrix

$$\underline{\underline{a}}^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \tag{1-7}$$

which is formed from the matrix $\underline{\underline{a}}$ by interchanging rows and columns, is known as the *transpose* of the matrix *a*.

The relations (1-3) and (1-4) can then be written†

$$\underline{\underline{a}}\underline{\underline{a}}^T = \underline{\underline{1}} \tag{1-3a}$$

$$\underline{\underline{a}}^T\underline{\underline{a}} = \underline{\underline{1}} \tag{1-4a}$$

or

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1-3b}$$

$$\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1-4b}$$

† Any matrix *b* which satisfies the relations $\underline{\underline{a}}\underline{\underline{b}} = \underline{\underline{b}}\underline{\underline{a}} = \underline{\underline{1}}$ is called the reciprocal of the matrix $\underline{\underline{a}}$ and denoted by $\underline{\underline{a}}^{-1}$. Then, according to (1-3*a*) and (1-4*a*), $\underline{\underline{a}}^T = \underline{\underline{a}}^{-1}$. Such a matrix whose transpose is equal to its reciprocal is known as an *orthogonal* matrix.

Equations (1-3) and (1-3*b*) or (1-4) and (1-4*b*) define the multiplication of two matrices. The matrix

$$\underline{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1-8}$$

is known as the *unit matrix*. Its elements are the Kronecker deltas δ_{ij}.

In recapitulation, if the representative of a physical quantity in any cartesian coordinate system has three components and these components transform according to the laws (1-1) and (1-2), the physical quantity is a vector. In what follows we shall usually consider the representation of a vector **A** in only a single coordinate system. In such a case we shall not distinguish between the vector **A** and its representative and use the notation **A** for both. Furthermore, we shall refer to the components of the representative as the components of **A**.

Consider two vectors **A** and **B** whose representatives in a given coordinate system, $\vec{A}$ and $\vec{B}$, have components A_1, A_2, A_3 and B_1, B_2, B_3, respectively. The vector **A** + **B** has for representative $\vec{A} + \vec{B}$ with components $A_1 + B_1$, $A_2 + B_2$, $A_3 + B_3$, while the vector **B** + **A** has for representative $\vec{B} + \vec{A}$ with components $B_1 + A_1$, $B_2 + A_2$, $B_3 + A_3$. Since the A_i and B_i are numbers, $B_i + A_i = A_i + B_i$. Hence

$$\vec{B} + \vec{A} = \vec{A} + \vec{B}$$

and **B** + **A** = **A** + **B**. This result expresses the *commutative law of vector addition;* namely, the sum of two vectors is independent of the order of addition.

In a like manner, it can be shown that

$$\vec{A} + \vec{B} + \vec{C} = (\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C}) = \vec{B} + (\vec{C} + \vec{A})$$

so that

$$\mathbf{A} + \mathbf{B} + \mathbf{C} = (\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}) = \mathbf{B} + (\mathbf{C} + \mathbf{A})$$

which expresses the *associative law of vector addition.*

1-3. VECTOR COMPONENTS

If we restrict ourselves to orthogonal coordinate systems, the component of a vector in a given direction is simply the projection of the vector along a line in that direction. The magnitude of the component is equal to the magnitude of the vector multiplied by the cosine of the angle between the vector and the given direction.

The component of **A** along the x_1 axis, for example, is A_1 where

$$A_1 = |\mathbf{A}| \cos(\mathbf{A},x_1) = A \cos(\mathbf{A},x_1) \tag{1-9}$$

where

$$A = |\mathbf{A}| = \sqrt{A_1^2 + A_2^2 + A_3^2} \tag{1-10}$$

denotes the magnitude of **A**, $(\mathbf{A},x_1)$ is the positive angle between **A** and the x_1 axis, and

$$\cos(\mathbf{A},x_1) = \frac{A_1}{A} \tag{1-11}$$

If $\mathbf{i}_1$, $\mathbf{i}_2$, and $\mathbf{i}_3$ denote unit vectors along the x_1, x_2, and x_3 axes, respectively, then

$$\mathbf{A} = A_1\mathbf{i}_1 + A_2\mathbf{i}_2 + A_3\mathbf{i}_3 \tag{1-12}$$

The component of **A** in any arbitrary direction **s** is

$$A_s = A \cos(\mathbf{A},\mathbf{s})$$

or, using (1-12),

$$\begin{aligned} A_s &= |A_1\mathbf{i}_1| \cos(\mathbf{i}_1,\mathbf{s}) + |A_2\mathbf{i}_2| \cos(\mathbf{i}_2,\mathbf{s}) + |A_3\mathbf{i}_3| \cos(\mathbf{i}_3,\mathbf{s}) \\ &= A_1 \cos(x_1,\mathbf{s}) + A_2 \cos(x_2,\mathbf{s}) + A_3 \cos(x_3,\mathbf{s}) \end{aligned} \tag{1-13}$$

1-4. THE SCALAR PRODUCT

It was demonstrated in the preceding section that the direction cosines of any vector **A** are given by A_1/A, A_2/A, A_3/A. Let the direction cosines of any other vector **B** be B_1/B, B_2/B, B_3/B. Then the cosine of the angle between **A** and **B**, $\theta = (\mathbf{A},\mathbf{B})$, is given by (see any text on analytic geometry)

$$\cos\theta = \frac{A_1}{A}\frac{B_1}{B} + \frac{A_2}{A}\frac{B_2}{B} + \frac{A_3}{A}\frac{B_3}{B}$$

Therefore

$$AB\cos\theta = A_1B_1 + A_2B_2 + A_3B_3 \tag{1-14}$$

The quantity $AB\cos\theta$ is known as the scalar product of the vectors **A** and **B** and is denoted by $\mathbf{A} \cdot \mathbf{B}$. Because of this notation the scalar product is also known as the dot product.

It follows from (1-14) that

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} = AB \cos(\mathbf{A},\mathbf{B}) \tag{1-15}$$

which shows that the scalar product obeys the commutative law of multiplication.

Application of (1-15) to the unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$, and $\mathbf{i}_3$ yields

$$\begin{aligned} &\mathbf{i}_1 \cdot \mathbf{i}_1 = \mathbf{i}_2 \cdot \mathbf{i}_2 = \mathbf{i}_3 \cdot \mathbf{i}_3 = 1 \\ &\mathbf{i}_1 \cdot \mathbf{i}_2 = \mathbf{i}_2 \cdot \mathbf{i}_1 = \mathbf{i}_2 \cdot \mathbf{i}_3 = \mathbf{i}_3 \cdot \mathbf{i}_2 = \mathbf{i}_3 \cdot \mathbf{i}_1 = \mathbf{i}_1 \cdot \mathbf{i}_3 = 0 \end{aligned} \tag{1-16}$$

It is left as an exercise for the reader to verify that

$$(\mathbf{C} + \mathbf{D}) \cdot \mathbf{B} = \mathbf{C} \cdot \mathbf{B} + \mathbf{D} \cdot \mathbf{B}$$

which states that the distributive law holds for the scalar product.

The scalar product of two vectors is an example of the physical quantity called a *scalar*. By definition, a scalar quantity is one whose representative in any cartesian coordinate system is a single number which remains invariant in a transformation of coordinates. In order to verify that the scalar product is, indeed, a scalar quantity, consider two vectors **A** and **B** whose representatives in two cartesian coordinate systems are $\vec{A}$, $\vec{B}$ and $\vec{A}'$, $\vec{B}'$, respectively. Then, using (1-2*a*),

$$\begin{aligned} \vec{A} \cdot \vec{B} &= \sum_i A_i B_i = \sum_i \Big(\sum_j a_{ji} A'_j\Big) \Big(\sum_k a_{ki} B'_k\Big) \\ &= \sum_{j,k} \Big(\sum_i a_{ji} a_{ki}\Big) A'_j B'_k = \sum_{j,k} \delta_{jk} A'_j B'_k \\ &= \sum_j A'_j B'_j = \vec{A}' \cdot \vec{B}' \end{aligned}$$

Thus, the scalar product of two vectors remains invariant in the transformation of coordinates and, therefore, is a scalar quantity.

1-5. THE VECTOR PRODUCT

The vector product of any two vectors **A** and **B** is denoted by $\mathbf{A} \times \mathbf{B}$. Because of this notation it is also known as the cross product. By definition, it is a vector perpendicular to the plane of the vectors **A** and **B** whose magnitude is equal to the product of the magnitudes of **A** and **B** by the sine of the angle between them and whose direction is that of the advance of a right-hand screw rotating from the first vector to the second through the smaller angle between their positive directions. Thus

$$|\mathbf{A} \times \mathbf{B}| = AB \sin(\mathbf{A},\mathbf{B}) \tag{1-17}$$

and

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \tag{1-18}$$

Equation (1-18) illustrates the failure of the commutative law of multiplication in the case of the vector product.

Application of the definition of the vector product to the unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$, and $\mathbf{i}_3$ yields

$$\begin{aligned} \mathbf{i}_1 \times \mathbf{i}_2 &= -\mathbf{i}_2 \times \mathbf{i}_1 = \mathbf{i}_3 \qquad & \mathbf{i}_2 \times \mathbf{i}_3 &= -\mathbf{i}_3 \times \mathbf{i}_2 = \mathbf{i}_1 \\ \mathbf{i}_3 \times \mathbf{i}_1 &= -\mathbf{i}_1 \times \mathbf{i}_3 = \mathbf{i}_2 \qquad & \mathbf{i}_1 \times \mathbf{i}_1 &= \mathbf{i}_2 \times \mathbf{i}_2 = \mathbf{i}_3 \times \mathbf{i}_3 = 0 \end{aligned} \tag{1-19}$$

It is left as an exercise for the reader to verify that

$$(\mathbf{C} + \mathbf{D}) \times \mathbf{B} = \mathbf{C} \times \mathbf{B} + \mathbf{D} \times \mathbf{B}$$

which states that the distributive law holds for the vector product. It then follows, using (1-19), that

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= (A_1\mathbf{i}_1 + A_2\mathbf{i}_2 + A_3\mathbf{i}_3) \times (B_1\mathbf{i}_1 + B_2\mathbf{i}_2 + B_3\mathbf{i}_3) \\ &= \mathbf{i}_1(A_2B_3 - A_3B_2) + \mathbf{i}_2(A_3B_1 - A_1B_3) + \mathbf{i}_3(A_1B_2 - A_2B_1) \end{aligned}$$

which can be expressed in the more compact determinant form

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} \tag{1-20}$$

It is interesting to note that the failure of the commutative law of multiplication in the case of the vector product now appears as a consequence of the fact that interchanging two rows of a determinant changes its sign. The vanishing of the vector product of two vectors in the same direction now appears as a consequence of the fact that a determinant vanishes if one of its rows is a multiple of another.

Because the vector product of two vectors differs from the vectors which we have considered thus far in one important respect, it is sometimes called a *pseudovector*. Consider two vectors **A** and **B**. Let $\vec{A}$ and $\vec{B}$ denote their representatives in one cartesian coordinate system and $\vec{A}'$ and $\vec{B}'$ their rèpresentatives in another system. Then, using (1-1*a*) and (1-2*a*),

$$\begin{aligned} (\vec{A} \times \vec{B})_1 = A_2B_3 - A_3B_2 &= \sum_{j,k} (a_{j2}a_{k3} - a_{j3}a_{k2}) A'_jB'_k \\ &= \sum_{j>k} (a_{j2}a_{k3} - a_{j3}a_{k2})(A'_jB'_k - A'_kB'_j) \\ &= \pm \sum_j a_{j1}(\vec{A}' \times \vec{B}')_j \end{aligned}$$

Thus, in general

$$(\vec{A} \times \vec{B})_i = \pm \sum_j a_{ji}(\vec{A}' \times \vec{B}')_j$$

The plus and minus signs arise from the sign of the determinant of the matrix of the direction cosines [cf. Eq. (1-5)]. The plus sign holds if the coordinate systems are both right-handed or both left-handed; the minus sign holds if one is right-handed and the other left-handed.

Clearly, the representatives of vectors and pseudovectors obey different laws for transformations between right-handed and left-handed systems. The inversion $x_1' = -x_1$, $x_2' = -x_2$, $x_3' = -x_3$ is an example of such transformations. It can be used in a simple method for distinguishing between vectors and pseudovectors. Consider any vector **A**. Let its representative in a given cartesian coordinate system be $\vec{A}$ with components (A_1, A_2, A_3). In the inverse coordinate system its representative is $\vec{A}'$ with components $(-A_1, -A_2, -A_3)$. The components of the representative of a vector change sign in an inversion of coordinates. Consider now two vectors **A** and **B**. Let their representatives in the given coordinate system and the inverse coordinate system be $\vec{A}$, $\vec{B}$ and $\vec{A}'$, $\vec{B}'$, respectively. Then

$$(\vec{A}' \times \vec{B}')_1 = A_2'B_3' - A_3'B_2' = A_2B_3 - A_3B_2 = (\vec{A} \times \vec{B})_1$$

with similar expressions for the other two components. Thus, the components of the representative of a vector product do not change sign on an inversion of coordinates.

1-6. THE TRIPLE SCALAR PRODUCT

Consider three arbitrary vectors **A**, **B**, and **C**. By definition, the triple scalar product is $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}$. Application of the results of the last two sections then yields

$$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \tag{1-21}$$

It follows at once on interchanging the position of the rows of the determinant that

$$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{B} \cdot \mathbf{C} \times \mathbf{A} = \mathbf{C} \cdot \mathbf{A} \times \mathbf{B} \tag{1-22}$$

There is then no ambiguity in writing the triple scalar product as **ABC**; i.e., the positions of the dot and cross are immaterial, and the three factors can be permuted cyclically without changing the product.

It should be noted that, while the scalar product of two vectors has the same representative in all cartesian coordinate systems, the representative of the triple scalar product (which is the scalar product of a vector and a pseudovector) changes sign in transformations between right-handed and left-handed coordinate systems. For this reason the triple scalar product is sometimes called a *pseudoscalar*.

1-7. THE TRIPLE VECTOR PRODUCT

This product is defined as $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$, where the parentheses indicate that one first performs the vector product of **B** with **C** and then the vector product of **A** with the resulting vector. In general

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \neq \mathbf{A} \times (\mathbf{B} \times \mathbf{C})$$

i.e., the triple vector product does not obey the associative law.

Let $\mathbf{F} = \mathbf{A} \times (\mathbf{B} \times \mathbf{C})$. Then from the definition of vector product, **F** is perpendicular to $(\mathbf{B} \times \mathbf{C})$ and must lie in the plane of **B** and **C**. Consequently

$$\mathbf{F} = \alpha\mathbf{B} + \beta\mathbf{C} \tag{1-23}$$

where α and β are scalar multipliers to be determined. The scalar product of **A** and **F** yields

$$\mathbf{A} \cdot \mathbf{F} = \alpha\mathbf{A} \cdot \mathbf{B} + \beta\mathbf{A} \cdot \mathbf{C} = 0$$

or

$$\frac{\alpha}{\mathbf{A} \cdot \mathbf{C}} = -\frac{\beta}{\mathbf{A} \cdot \mathbf{B}} \tag{1-24}$$

since **F** is perpendicular to **A**. Then

$$\alpha = n(\mathbf{A} \cdot \mathbf{C}) \qquad \text{and} \qquad \beta = -n(\mathbf{A} \cdot \mathbf{B}) \tag{1-25}$$

where n is some constant scalar to be determined.

Substitution of (1-25) into (1-23) yields

$$\mathbf{F} = n[(\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}] \tag{1-26}$$

the x_1 component of which is

$$F_1 = n[(A_1C_1 + A_2C_2 + A_3C_3)B_1 - (A_1B_1 + A_2B_2 + A_3B_3)C_1] \tag{1-27}$$

The x_1 component of $\mathbf{F} = \mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ is given by

$$\begin{aligned} F_1 &= A_2(\mathbf{B} \times \mathbf{C})_3 - A_3(\mathbf{B} \times \mathbf{C})_2 \\ &= A_2(B_1C_2 - B_2C_1) - A_3(B_3C_1 - B_1C_3) \\ &= (A_1C_1 + A_2C_2 + A_3C_3)B_1 - (A_1B_1 + A_2B_2 + A_3B_3)C_1 \end{aligned} \tag{1-28}$$

Comparison of Eqs. (1-27) and (1-28) shows that $n = 1$ and, consequently, that

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} \tag{1-29}$$

1-8. THE GRADIENT

Let a scalar ϕ be associated with every point (x_1,x_2,x_3) of space in such a way that $\phi(x_1,x_2,x_3)$ is a continuous and differentiable function of position. In the transition from the point (x_1,x_2,x_3) to the point $(x_1 + dx_1,\ x_2 + dx_2,\ x_3 + dx_3)$ the scalar ϕ undergoes a change

$$d\phi = \frac{\partial \phi}{\partial x_1} dx_1 + \frac{\partial \phi}{\partial x_2} dx_2 + \frac{\partial \phi}{\partial x_3} dx_3 \tag{1-30}$$

corresponding to the change

$$d\mathbf{s} = dx_1\,\mathbf{i}_1 + dx_2\,\mathbf{i}_2 + dx_3\,\mathbf{i}_3 \tag{1-31}$$

in the position vector **s**.

The right-hand member of Eq. (1-30) may be considered as the scalar product of $d\mathbf{s}$ with the vector

$$\text{grad}\ \phi = \frac{\partial \phi}{\partial x_1}\mathbf{i}_1 + \frac{\partial \phi}{\partial x_2}\mathbf{i}_2 + \frac{\partial \phi}{\partial x_3}\mathbf{i}_3 \tag{1-32}$$

where grad ϕ is read "gradient of ϕ." Thus

$$d\phi = \text{grad}\ \phi \cdot d\mathbf{s} \tag{1-33}$$

Since the scalar product of grad ϕ and $d\mathbf{s}$ is zero when $d\mathbf{s}$ lies in the surface ϕ = constant, it follows that grad ϕ is perpendicular to this surface. Such a surface for which ϕ is a constant is known as an *equipotential surface.*

The directional derivative of the scalar ϕ in any direction **s** is

$$\frac{d\phi}{ds} = \frac{\text{grad}\ \phi \cdot d\mathbf{s}}{ds} = \text{grad}\ \phi \cdot \mathbf{i}_s \tag{1-34}$$

where $\mathbf{i}_s$ is a unit vector in the direction of **s**. Clearly, the maximum value of $d\phi/ds$ is $|\text{grad}\ \phi|$. Thus grad ϕ represents in magnitude and direction the greatest space rate of change of ϕ.

1-9. CONSERVATIVE VECTOR FIELD

By definition, a conservative vector field is a vector field for which the line integral of the vector about any closed path is zero. Thus, if the vector be denoted by **F**,

$$\oint \mathbf{F} \cdot d\mathbf{s} = 0 \tag{1-35}$$

where $d\mathbf{s}$ is an element of the path and the symbol $\oint$ indicates a line integral about a closed path.

A necessary and sufficient condition for a conservative vector field is that the vector be expressible as the gradient of a scalar. This fact can be demonstrated as follows: Let $\mathbf{F} = -\text{grad}\ \phi$. Then

$$\int_a^b \mathbf{F} \cdot d\mathbf{s} = -\int_a^b \text{grad}\ \phi \cdot d\mathbf{s}$$
$$= -\int_a^b d\phi = \phi(a) - \phi(b) \tag{1-36}$$

From (1-36) it is evident that the line integral of the gradient of a scalar between any two points a and b is independent of the path of integration, depending solely on the value of the scalar at the initial and final points. Hence $\oint \text{grad}\ \phi \cdot d\mathbf{s} = 0$.

According to (1-36) the value of the scalar ϕ at any point b in terms of its value at any point a is

$$\phi(b) = \phi(a) - \int_a^b \mathbf{F} \cdot d\mathbf{s} \tag{1-37}$$

where the path of integration is entirely arbitrary. Now let the coordinates of the point a be (x_1,x_2,x_3) and those of b be $(x_1 + dx_1,\ x_2 + dx_2,\ x_3 + dx_3)$. Then

$$\phi(b) - \phi(a) = d\phi = \text{grad}\ \phi \cdot d\mathbf{s}$$

where $d\mathbf{s} = dx_1\,\mathbf{i}_1 + dx_2\,\mathbf{i}_2 + dx_3\,\mathbf{i}_3$, and (1-37) becomes

$$\text{grad}\ \phi \cdot d\mathbf{s} = -\mathbf{F} \cdot d\mathbf{s}$$

provided that $\mathbf{F}$ is at least piece-wise continuous. Since $d\mathbf{s}$ is arbitrary, it follows that $\mathbf{F} = -\text{grad}\ \phi$.

1-10. DIVERGENCE OF A VECTOR

It was seen in Sec. 1-8 that

$$\text{grad}\ \phi = \frac{\partial \phi}{\partial x_1}\mathbf{i}_1 + \frac{\partial \phi}{\partial x_2}\mathbf{i}_2 + \frac{\partial \phi}{\partial x_3}\mathbf{i}_3 \tag{1-32}$$

It is now convenient to rewrite the expression as

$$\text{grad}\ \phi = \left(\frac{\partial}{\partial x_1}\mathbf{i}_1 + \frac{\partial}{\partial x_2}\mathbf{i}_2 + \frac{\partial}{\partial x_3}\mathbf{i}_3\right)\phi$$

where the quantity in the parentheses is a vector operator known as the *del operator* and denoted by ∇. Consequently,

$$\text{grad}\ \phi \equiv \nabla\phi$$

Application of the rule of scalar multiplication to the vectors

$$\nabla = \frac{\partial}{\partial x_1}\mathbf{i}_1 + \frac{\partial}{\partial x_2}\mathbf{i}_2 + \frac{\partial}{\partial x_3}\mathbf{i}_3$$

and

$$\mathbf{F} = F_1\mathbf{i}_1 + F_2\mathbf{i}_2 + F_3\mathbf{i}_3$$

in the order $\nabla \cdot \mathbf{F}$ yields a scalar known as the divergence of **F** and written as

$$\nabla \cdot \mathbf{F} = \operatorname{div} \mathbf{F} = \frac{\partial}{\partial x_1}F_1 + \frac{\partial}{\partial x_2}F_2 + \frac{\partial}{\partial x_3}F_3 \tag{1-38}$$

The above treatment has been purely formal. The physical significance of the divergence of any vector will become apparent in the consideration of Gauss' theorem (Sec. 1-15).

Note that, since the vector ∇ is an operator,

$$\nabla \cdot \mathbf{F} \neq \mathbf{F} \cdot \nabla = F_1\frac{\partial}{\partial x_1} + F_2\frac{\partial}{\partial x_2} + F_3\frac{\partial}{\partial x_3} \tag{1-39}$$

The application of the rule of scalar multiplication to the vectors ∇ and **F** in the order $\mathbf{F} \cdot \nabla$ yields a new *scalar operator.*

1-11. CURL OF A VECTOR

It was seen in the preceding section that the scalar product $\nabla \cdot \mathbf{F}$ of the del operator ∇ and any vector **F** gave rise to the scalar-point function called the divergence of the vector **F**. Similarly, the vector product of ∇ and **F** in the order $\nabla \times \mathbf{F}$ gives rise to a vector-point function known as the curl of the vector **F** and written

$$\nabla \times \mathbf{F} = \operatorname{curl} \mathbf{F} = \mathbf{i}_1\left(\frac{\partial}{\partial x_2}F_3 - \frac{\partial}{\partial x_3}F_2\right) + \mathbf{i}_2\left(\frac{\partial}{\partial x_3}F_1 - \frac{\partial}{\partial x_1}F_3\right) + \mathbf{i}_3\left(\frac{\partial}{\partial x_1}F_2 - \frac{\partial}{\partial x_2}F_1\right)$$

$$= \begin{vmatrix} \mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ F_1 & F_2 & F_3 \end{vmatrix} \tag{1-40}$$

Note that, since the vector ∇ is an operator, $\nabla \times \mathbf{F} \neq -\mathbf{F} \times \nabla$, which is another vector operator.

The vanishing of the curl of a vector at all points in space is a necessary condition which any vector **F** must satisfy if it is derivable from a potential. Thus, if $\mathbf{F} = -\operatorname{grad} \phi$, then $\operatorname{curl} \mathbf{F} = 0$. The proof is as follows: Since $\mathbf{F} = -\operatorname{grad} \phi$,

$$\int \mathbf{F} \cdot d\mathbf{s} = -\int \operatorname{grad} \phi \cdot d\mathbf{s}$$