Higher Mathematics
for
Engineers and Physicists

BY

IVAN S. SOKOLNIKOFF, Ph.D.
Professor of Mathematics, University of Wisconsin

AND

ELIZABETH S. SOKOLNIKOFF, Ph.D.
Lecturer in Mathematics, University of Wisconsin

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PREFACE

The favorable reception of the First Edition of this volume appears to have sustained the authors' belief in the need of a book on mathematics beyond the calculus, written from the point of view of the student of applied science. The chief purpose of the book is to help to bridge the gap which separates many engineers from mathematics by giving them a bird's-eye view of those mathematical topics which are indispensable in the study of the physical sciences.

It has been a common complaint of engineers and physicists that the usual courses in advanced calculus and differential equations place insufficient emphasis on the art of formulating physical problems in mathematical terms. There may also be a measure of truth in the criticism that many students with pronounced utilitarian leanings are obliged to depend on books that are more distinguished for rigor than for robust uses of mathematics.

This book is an outgrowth of a course of lectures offered by one of the authors to students having a working knowledge of the elementary calculus. The keynote of the course is the practical utility of mathematics, and considerable effort has been made to select those topics which are of most frequent and immediate use in applied sciences and which can be given in a course of one hundred lectures. The illustrative material has been chosen for its value in emphasizing the underlying principles rather than for its direct application to specific problems that may confront a practicing engineer.

In preparing the revision the authors have been greatly aided by the reactions and suggestions of the users of this book in both academic and engineering circles. A considerable portion of the material contained in the First Edition has been rearranged and supplemented by further illustrative examples, proofs, and problems. The number of problems has been more than doubled. It was decided to omit the discussion of improper integrals and to absorb the chapter on Elliptic Integrals into much enlarged chapters on Infinite Series and Differential Equations. A new chapter on Complex Variable incorporates some of the material that was formerly contained in the chapter on Conformal Representation. The original plan of making each chapter as nearly as possible an independent unit, in order to provide some flexibility and to enhance the availability of the book for reference purposes, has been retained.

I. S. S.
E. S. S.

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HIGHER MATHEMATICS FOR ENGINEERS AND PHYSICISTS

CHAPTER I

INFINITE SERIES

It is difficult to conceive of a single mathematical topic that occupies a more prominent place in applied mathematics than the subject of infinite series. Students of applied sciences meet infinite series in most of the formulas they use, and it is quite essential that they acquire an intelligent understanding of the concepts underlying the subject.

The first section of this chapter is intended to bring into sharper focus some of the basic (and hence more difficult) notions with which the reader became acquainted in the first course in calculus. It is followed by ten sections that are devoted to a treatment of the algebra and calculus of series and that represent the minimum theoretical background necessary for an intelligent use of series. Some of the practical uses of infinite series are indicated briefly in the remainder of the chapter and more fully in Chaps. II, VII, and VIII.

1. Fundamental Concepts. Familiarity with the concepts discussed in this section is essential to an understanding of the contents of this chapter.

FUNCTION. *The variable y is said to be a function of the variable x if to every value of x under consideration there corresponds at least one value of y.*

If x is the variable to which values are assigned at will, then it is called the *independent variable.* If the values of the variable y are determined by the assignment of values to the independent variable x, then y is called the *dependent variable.* The functional dependence of y upon x is usually denoted by the equation*

$$y = f(x).$$

Unless a statement to the contrary is made, it will be supposed in this book that the variable x is permitted to assume real values only and that the corresponding values of y are also real. In this event the function $f(x)$ is called a *real function of the real variable x.* It will be observed that

$$y = \sqrt{x} \tag{1-1}$$

does not represent a real function of x for all real values of x, for the values of y become imaginary if x is negative. In order that the symbol $f(x)$ define a real function of x, it may be necessary to restrict the range of values that x may assume. Thus, (1-1) defines a real function of x only if $x \geq 0$. On the other hand, $y = \sqrt{x^2 - 1}$ defines a real function of x only if $|x| \geq 1$.

SEQUENCES AND LIMITS. Let some process of construction yield a succession of values

$$x_1, x_2, x_3, \cdots, x_n, \cdots,$$

where it is assumed that every x_i is followed by other terms. Such a succession of terms is called an *infinite sequence.* Examples of sequences are

(a) $1, 2, 3, \cdots, n, \cdots,$

(b) $\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \cdots, (-1)^{n-1}\frac{1}{2^n}, \cdots,$

(c) $0, 2, 0, 2, \cdots, 1 + (-1)^n, \cdots.$

Sequences will be considered here only in connection with the theorems on infinite series,† and for this purpose it is necessary to have a definition of the limit of a sequence.

DEFINITION. *The sequence $x_1, x_2, \cdots, x_n, \cdots$ is said to converge to the constant L as a limit if for any preassigned positive number ϵ, however small, one can find a positive integer p such that*

$$|x_n - L| < \epsilon \qquad \text{for all } n > p.$$

* Other letters are often used. In particular, if more than one function enters into the discussion, the functions may be denoted by $f_1(x)$, $f_2(x)$, etc.; by $f(x)$, $g(x)$, etc.; by $F(x)$, $G(x)$, etc.

† For a somewhat more extensive treatment, see I. S. Sokolnikoff, Advanced Calculus, pp. 3–21.

For convenience, this definition is frequently written in the compact form

$$\lim_{n \to \infty} x_n = L,$$

and L is called the *limit* of the sequence. If a variable x takes on these successive values $x_1, x_2, \cdots, x_n, \cdots$, then x is said to *approach L as a limit.* It follows from this definition that, of the sequences given above, (*b*) converges to the limit 0, whereas (*a*) and (*c*) are not convergent.

As an illustration, let the variable x assume the set of values

$$x_1 = 0.1, \qquad x_2 = 0.11, \qquad x_3 = 0.111, \cdots.$$

It is easily seen that

$$\lim_{n \to \infty} x_n = \frac{1}{9};$$

that is, corresponding to any $\epsilon > 0$, one can find a positive integer p such that

$$|L - x_n| \equiv \left|\frac{1}{9} - x_n\right| < \epsilon$$

for all values of n greater than p. Observe that

$$\frac{1}{9} - x_1 = \frac{1}{90}, \qquad \frac{1}{9} - x_2 = \frac{1}{900}, \quad \cdots, \qquad \frac{1}{9} - x_n = \frac{1}{9 \cdot 10^n}.$$

Hence, for any ϵ that is chosen, it is necessary to demand that n be large enough so that

$$\frac{1}{9} - x_n \equiv \frac{1}{9 \cdot 10^n} < \epsilon.$$

The inequality is equivalent to

$$9 \cdot 10^n > \frac{1}{\epsilon},$$

and, taking logarithms to the base 10,*

$$\log 9 + n > \log \frac{1}{\epsilon}$$

or

$$n > -(\log 9 + \log \epsilon) = -\log 9\epsilon.$$

* From the definition of the logarithm, it follows that, if $A > B$, then $\log A > \log B$.

Thus, if p is chosen as any integer greater than $|-\log 9\epsilon|$, the inequality

$$\left|\frac{1}{9} - x_n\right| < \epsilon$$

will be satisfied for all values of n greater than p.

INFINITE SERIES. Let $u_1, u_2, u_3, \cdots$ be an infinite sequence of real functions of a real variable x. Then the symbol

$$(1\text{-}2) \qquad \sum_{n=1}^{\infty} u_n(x) \equiv u_1(x) + u_2(x) + \cdots + u_n(x) + \cdots$$

is called an *infinite series.*

If, in (1-2), x is assigned some fixed value, say $x = x_0$, there results the series of constants

$$(1\text{-}3) \qquad \sum_{n=1}^{\infty} u_n(x_0).$$

Denote by $s_n(x_0)$ the nth partial sum, that is, the sum of the first n terms, of the series (1-3) so that

$$s_n(x_0) = u_1(x_0) + u_2(x_0) + \cdots + u_n(x_0).$$

As n increases indefinitely, the sequence of constants

$$s_1(x_0),\ s_2(x_0),\ \cdots,\ s_n(x_0),\ \cdots$$

either will converge to a finite limit S or it will not converge to such a limit. If

$$\lim_{n\to\infty} s_n(x_0) = S,$$

the series (1-2) is said to converge to the value S for $x = x_0$.* If the series (1-2) converges for every value of x in some interval† (a, b), then the series is said to be convergent in the interval (a, b).

As an example, consider the series

$$(1\text{-}4) \qquad 1 + x + x^2 + \cdots + x^{n-1} + \cdots.$$

If $x = \frac{1}{2}$, (1-4) becomes

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} + \cdots,$$

* This limit S is usually called the *sum of the series* (1-3).

† This means that x can assume any real value between a and b and that a and b can be thought of as the end points of an interval of the x-axis.

which is convergent to the value 2. In order to establish this fact, note that

$$s_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}}$$

is a geometric progression of ratio ½, so that

$$s_n = \frac{1 - \dfrac{1}{2^n}}{1 - \dfrac{1}{2}} = 2 - \frac{1}{2^{n-1}}.$$

Hence, the absolute value of the difference between 2 and s_n is $1/2^{n-1}$, which can be made arbitrarily small by choosing n sufficiently large.

On the other hand, if $x = -1$, the series (1-4) becomes

$$1 - 1 + 1 - 1 + \cdots + (-1)^{n-1} + \cdots,$$

which does not converge; for $s_{2n} = 0$ and $s_{2n-1} = 1$ for any choice of n and, therefore, $\lim_{n\to\infty} s_n$ does not exist. Moreover, if $x = 2$, the series (1-4) becomes

$$1 + 2 + 4 + \cdots + 2^{n-1} + \cdots,$$

so that s_n increases indefinitely with n and $\lim_{n\to\infty} s_n$ does not exist. If an infinite series does not converge for a certain value of x, it is said to *diverge* or *be divergent* for that value of x. It will be shown later that the series (1-4) is convergent for $-1 < x < 1$ and divergent for all other values of x.

The definition of the limit, as given above, assumes that the value of the limit S is known. Frequently it is possible to infer the existence of S without actually knowing its value. The following example will serve to illustrate this point.

Example. Consider the series

$$s = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots,$$

and compare the sum of its first n terms

$$s_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

with the sum of the geometrical progression

$$S_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}$$
$$= 2 - \frac{1}{2^{n-1}}.$$

The corresponding terms of S_n are never less than those of s_n; but, no matter how large n be taken, S_n is less than 2. Consequently, $s_n < 2$; and since the successive values of s_n form an increasing sequence of numbers, the sum of the first series must be greater than 1 and less than or equal to 2. A geometrical interpretation of this statement may help to fix the idea. If the successive values of s_n,

$$s_1 = 1,$$
$$s_2 = 1 + \frac{1}{2!} = 1.5,$$
$$s_3 = 1 + \frac{1}{2!} + \frac{1}{3!} = 1.667,$$
$$s_4 = 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} = 1.708,$$
$$s_5 = 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} = 1.717,$$

are plotted as points on a straight line (Fig. 1), the points representing the sequence $s_1, s_2, \cdots, s_n, \cdots$ always move to the right but never

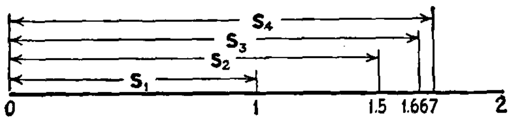

FIG. 1.

progress as far as the point 2. It is intuitively clear that there must be some point s, either lying to the left of 2 or else coinciding with it, which the numbers s_n approach as a limit. In this case the numerical value of the limit has not been ascertained, but its existence was established with the aid of what is known as the *fundamental principle.*

Stated in precise form the principle reads as follows: *If an infinite set of numbers* $s_1, s_2, \cdots, s_n, \cdots$ *forms an increasing sequence* (that is, $s_N > s_n$, *when* $N > n$) *and is such that every* s_n *is less than some fixed number* M (that is, $s_n < M$ *for all values of* n), *then* s_n *approaches a limit* s *that is not greater than* M (that is, $\lim_{n\to\infty} s_n = s \leq M$). The formulation of the principle for a decreasing sequence of numbers $s_1, s_2, \cdots, s_n, \cdots$, which are always greater than a certain fixed number m, will be left to the reader.

2. Series of Constants. The definition of the convergence of a series of functions evidently depends on a study of the behavior of series of constants. The reader has had some acquaintance with such series in his earlier study of mathematics, but it seems desirable to provide a summary of some essential theorems that will be needed later in this chapter. The following important theorem gives the necessary and sufficient condition for the convergence of an infinite series of constants:

THEOREM. *The infinite series of constants* $\sum_{n=1}^{\infty} u_n$ *converges if and only if there exists a positive integer n such that for all positive integral values of p*

$$|s_{n+p} - s_n| \equiv |u_{n+1} + u_{n+2} + \cdots + u_{n+p}| < \epsilon,$$

where ϵ *is any preassigned positive constant.*

The necessity of the condition can be proved immediately by recalling the definition of convergence. Thus, assume that the series converges, and let its sum be S, so that

$$\lim_{n\to\infty} s_n = S$$

and also, for any fixed value of p,

$$\lim_{n\to\infty} s_{n+p} = S.$$

Hence,

$$\lim_{n\to\infty} (s_{n+p} - s_n) \equiv \lim_{n\to\infty} (u_{n+1} + u_{n+2} + \cdots + u_{n+p}) = 0,$$

which is another way of saying that

$$|u_{n+1} + u_{n+2} + \cdots + u_{n+p}| < \epsilon$$

for a sufficiently large value of n.

The proof of the sufficiency of the condition requires a fair degree of mathematical maturity and will not be given here.*

This theorem is of great theoretical importance in a variety of investigations, but it is seldom used in any practical problem requiring the testing of a given series. A number of tests for convergence, applicable to special types of series, will be given in the following sections.

It may be remarked that a sufficient condition that a series diverge is that the terms u_n do not approach zero as a limit when n increases indefinitely. Thus the necessary condition for convergence of a series is that $\lim_{n\to\infty} u_n = 0$, but this condition is not

* See SOKOLNIKOFF, I. S., Advanced Calculus, pp. 11–13.

sufficient; that is, there are series for which $\lim_{n \to \infty} u_n = 0$ but which are not convergent. A classical example illustrating this case is the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots,$$

in which S_n increases without limit as n increases.

Despite the fact that a proof of the divergence of the harmonic series is given in every good course in elementary calculus, it will be recalled here because of its importance in subsequent considerations. Since

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} > n \cdot \frac{1}{2n} = \frac{1}{2},$$

it is possible, beginning with any term of the series, to add a definite number of terms and obtain a sum greater than $\frac{1}{2}$. If $n = 2$,

$$\frac{1}{3} + \frac{1}{4} > \frac{1}{2};$$

$n = 4$,

$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{2};$$

$n = 8$,

$$\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16} > \frac{1}{2};$$

$n = 16$,

$$\frac{1}{17} + \frac{1}{18} + \cdots + \frac{1}{32} > \frac{1}{2}.$$

Thus it is possible to group the terms of the harmonic series

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots$$

in such a way that the sum of the terms in each parenthesis exceeds $\frac{1}{2}$; and, since the series

$$1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

is obviously divergent, the harmonic series is divergent also.

3. Series of Positive Terms. This section is concerned with series of the type

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots,$$

where the a_n are positive constants. It is evident from the definition of convergence and from the fundamental principle (see Sec. 1) that the convergence of a series of positive constants will be established if it is possible to demonstrate that the partial sums s_n remain bounded. This means that there exists some positive number M such that $s_n < M$ for all values of n. The proof of the following important test is based on such a demonstration.

COMPARISON TEST. *Let* $\sum_{n=1}^{\infty} a_n$ *be a series of positive terms, and let* $\sum_{n=1}^{\infty} b_n$ *be a series of positive terms that is known to converge. Then the series* $\sum_{n=1}^{\infty} a_n$ *is convergent if there exists an integer* p *such that, for* $n \geq p$, $a_n \leq b_n$. *On the other hand, if* $\sum_{n=1}^{\infty} c_n$ *is a series of positive terms that is known to be divergent and if* $a_n \geq c_n$ *for* $n \geq p$, *then* $\sum_{n=1}^{\infty} a_n$ *is divergent also.*

Since the convergence or divergence of a series evidently is not affected by the addition or subtraction of a finite number of terms, the proof will be given on the assumption that $p = 1$. Let $s_n = a_1 + a_2 + \cdots + a_n$, and let B denote the sum of the series $\sum_{n=1}^{\infty} b_n$ and B_n its nth partial sum. Then, since $a_n \leq b_n$ for all values of n, it follows that $s_n \leq B_n$ for all values of n. Hence, the s_n remain bounded, and the series $\sum_{n=1}^{\infty} a_n$ is convergent. On the other hand, if $a_n \geq c_n$ for all values of n and if the series $\sum_{n=1}^{\infty} c_n$ diverges, then the series $\sum_{n=1}^{\infty} a_n$ will diverge also.

There are two series that are frequently used as series for comparison.

a. The geometric series

$$a + ar + ar^2 + \cdots + ar^n + \cdots, \tag{3-1}$$