Number Theory and Algebra

EDITED BY

HANS ZASSENHAUS

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Preface

We are happy to honor three well-known number theorists, Henry B. Mann, Arnold E. Ross, and Olga Taussky-Todd. We do this at a time when they can look back over many years of honest toil and productive work.

We, their colleagues, pupils, collaborators, and friends find it fitting to dedicate to them the fruits of our work so as to pass on to researchers coming after us the spirit of patient toil in the service of the queenly science which our three honorees implanted in us.

We would like to thank the publisher for encouraging the publication of this special volume. We are also grateful for the overwhelming response by the authors upon this happy occasion.

As a study of the table of contents of this book will suggest to the reader, there is an enormous variety and depth to the research efforts of the modern number theorist and algebraist. As a first guide to the reader who is familiar with the traditional classification of the fields under consideration, we would like to group the contents along the following lines.

Elementary number theory is one of the most ancient sources of number theoretical and algebraic speculation which only now seems to be able to find its true depth. See for example the contribution of "Erdös" type by P. Erdös and E. G. Strauss. Richard K. Guy's contribution has a more experimental flavor. Emma Lehmer follows in the steps of C. F. Gauss (1777–1855). An application of Lie algebra is made by J. W. B. Hughes.

Numbers as objects of *statistical analytic number theory* investigations are dealt with by K. Mahler, by R. L. Graham, H. S. Witsenhausen, and J. H. Spencer; also by Wolfgang M. Schmidt as well as by the late S. Knapowski and P. Turán.

An application of the *theory of modular forms* is given by Bram van Asch.

The *combinatorial aspects of number theory* and their connection with *group theory* are dealt with in the contribution by John E. Olson and Edward T. White; also by Sündar Lal, R. L. McFarland, and R. W. K. Odoni in response to a challenging problem of H. Mann on difference sets. The *combinatorial properties of finite fields* are dealt with by D. K. Ray-Chaudhuri and N. M. Singhi. An application of such ideas in *finite projective geometry* was made by Jill C. D. S. Yaqub.

The *algebraic theory of quadratic forms and their applications* is expounded by Warren Wolfe. A novel more abstract treatment of the theory is given by Manfred Knebusch.

The classical *arithmetical theory of quadratic forms* is dealt with in the articles by A. G. Earnest and J. S. Hsia, Robert Gold and Paul Ponomarev, and by Marshall Hall, Jr.

The *algebra of finite extensions* was enriched by the late Bohuslav Diviš in reply to a challenging question of A. Schinzl.

The *theory of algebraic number fields and their arithmetics* is richly represented by the contributions of D. S. Dummit and H. Kisilevsky, A. Fröhlich, Dennis A. Garbanati, Basil Gordon and Murray Schacher, Gary Queen, Ross Schipper, William Yslas Vélez, by Sunder Lal, R. L. McFarland, and R. W. K. Odoni, and by M. Pohst.

The principles of *noncommutative arithmetics* are dealt with by Wilhelm Plesken and by Hans Zassenhaus.

The *geometry of numbers*, an ally of number theoretical research since Gauss and Dirichlet, which was developed as a new mathematical science by Hermite and Minkowski, is represented by the article of R. P. Bambah and A. C. Woods.

The *algebraic geometry of curves* providing a rich field of investigation in the spirit of number theory is dealt with in the contribution of Manohar L. Madan and Sat Pal. An application in the same spirit on the *theory of formal power series* is given by Charles F. Osgood.

Biographical Sketches

Henry B. Mann

In 1975 Henry B. Mann celebrated his seventieth birthday. A mathematician of international fame, Mann, in a career of more than forty years, has made significant contributions to algebra, number theory, statistics, and combinatorics.

Henry Mann was born October 27, 1905, in Vienna. He received his Ph D. degree in mathematics in 1935 from the University of Vienna where, as a student of Philipp Furtwängler, he wrote his dissertation in algebraic number theory. After a year of teaching school in Vienna and a couple of years spent in research and tutoring, he emigrated in 1938 to the United States.

In New York he earned his living for several years primarily by tutoring. He had by then developed an interest in mathematical statistics, particularly in the analysis of variance, and in the problem of designing experiments with a view to their statistical analysis. He later contributed to this subject in a number of research papers and in his book (1949) "Analysis and Design of Experiments."

One of Mann's most remarkable achievements was his discovery in 1941 of a proof of a celebrated conjecture of Schnirelmann and Landau in additive number theory. This conjecture had its origin in the work of L. Schnirelmann in the early 1930s. Schnirelmann had considered a density $\delta(A)$ for a set of positive integers A, which he defined by

$$\delta(A) = \inf\{A(n)/n \mid n = 1, 2, \ldots\},$$

where $A(n)$ is the number of positive integers $\leq n$ in the set A. He showed (most easily) that if the sum $A + B$ of two sets of positive integers is formed by $A + B = \{a, b, a + b | a \in A, b \in B\}$, then the density satisfies the rules:

(1) $\delta(A + B) \geq \delta(A) + \delta(B) - \delta(A)\delta(B)$.
(2) $A + B$ contains all positive integers if $\delta(A) + \delta(B) \geq 1$.

From these two rules he obtained (readily) the result that *any set having positive density is a basis for the integers* (that is, if $\delta(A) > 0$, then the sum of A with itself sufficiently many times contains all positive integers). As an application of these ideas, Schnirelmann proved (for the first time) the existence of a value k such that every integer greater than 1 is the sum of at most k primes. This he did by showing that $P + P$—P is the set of primes together with 1—has positive density, hence is a basis for the integers.

Out of further study of these ideas by Schnirelmann himself and by E. Landau, there arose the conjecture that (1) and (2) may be replaced by the much stronger statement: *Either $A + B$ contains all positive integers or*

$$\delta(A + B) \geq \delta(A) + \delta(B).$$

This conjecture, appealing in its apparent simplicity, soon attracted wide attention. Many distinguished mathematicians attempted to find a proof; indeed, partial results were obtained over the next decade by E. Landau, A. Khintchine, A. Besicovitch, I. Schur, and A. Brauer.

It was this conjecture that Mann succeeded in proving in 1941. His interest in the problem had been aroused through the lectures of A. Brauer at New York University. Actually, he proved the still sharper statement: *If $C = A + B$, then either $C(n) = n$ or*

$$\frac{C(n)}{n} \geq \min \frac{A(m) + B(m)}{m}, \qquad 1 \leq m \leq n, \quad m \notin C.$$

For his proof he was awarded the Cole Prize in Number Theory by the American Mathematical Society in 1946. The technique that Mann introduced in his proof, and its various modifications, have led to further important results in additive number theory and have also proved useful in the more general setting of additive problems in groups.

In 1942 Mann was the recipient of a Carnegie Fellowship for the study of statistics at Columbia University. At Columbia he had the opportunity of working with Abraham Wald in the department of economics, which at that time was headed by Harold Hotelling. He taught for a year (1943–1944) in the Army Specialized Training Program at Bard College; he spent a year (1944–1945) as research associate at Ohio State University, and six months as research associate at Brown University. In 1946 he returned to Ohio State to join the mathematics faculty where, as associate professor (1946–1948) and full professor (1948–1964), he was actively engaged in teaching and research for many years. He held professorships at the University of Wisconsin Mathematics Research Center from 1964 to 1971, and at the University of Arizona from 1971 until his retirement in 1975.

Mann's research interests in algebra and combinatorics cover a wide range. He has a special fondness, though, for algebraic number theory and Galois theory, and has imparted his enthusiasm for these subjects to many students over the years. Besides his dozen or so papers that contribute directly to these subjects, several of his papers on difference sets and coding theory contain beautiful applications of theorems on algebraic numbers and Galois theory.

Neither Henry Mann's research nor, for that matter, his teaching have ended with his retirement; he spent the past spring term as visiting professor at Oregon State University. He and his wife Anne, who this year observed their 41st wedding anniversary, plan to continue living in Tucson, Arizona.

His friends, colleagues, and students wish him many more years of good health and enjoyment of mathematics.

JOHN OLSON

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Arnold E. Ross

The following is a quotation from an address in honor of
Arnold E. Ross
at the Meeting of the Ohio Section of the Mathematical Association of America
at Youngstown, Ohio, May 7, 1976

Tonight we are here to honor Arnold E. Ross, the man, the mathematician, and the leader of men.

We are very glad to have Arnold and Bee with us. She has been his companion in everyday's perplexities and his pillar of strength.

Born in Chicago in 1906 to an immigrant couple from southern Russia, Arnold was taken by his mother back to the old country in his early youth for a visit with his grandparents. The first World War and then the revolution cut them off from the USA for many years. He received most of his precollege education in Odessa, a cultured city on the Black Sea which provided the early environment of many famous scientists, musicians, and politicians.

Early influences are most formative and of the greatest impact on the intellectual development of a person and on the direction of his life work.

Arnold must have been a precocious youngster since he was able to enter the University under a special arrangement at less than sixteen years of age. It was Professor Shatunovsky who was deeply interested in very gifted youngsters and who was a friend of Arnold Ross' physician uncle who encouraged this early start and helped with the needed arrangements. Besides Arnold, a cousin of his as well as one of his friends and another future mathematician, Felix Gantmacher, took part in the program, all of about the same boyish age. S. O. Shatunovsky (1859–1929), a distinguished Russian mathematician and educator, was one of the leaders of science and mathematics education in postrevolutionary Russia. Another prominent leader of science education in the 1930s in Russia was the energetic physicist Peter Kapitza (1894b.).

A small number of farsighted distinguished Russian scientists and mathematicians building on a centuries old excellent academic tradition succeeded in the twenties and thirties in designing and implementing a broadly based, competitive system of science and mathematics education in the elementary and secondary schools of the USSR, succeeding thereby in creating many more opportunities for the talented children of all sectors of the people than there had been available in czarist Russia.

In this country, Arnold's summer program for gifted high school students and a few other similarly oriented programs amply demonstrated in the sixties and seventies that a sizeable number of boys and girls at an age of 13 or slightly older are at the peak of their learning power, full of curiosity and eagerness to explore new avenues of thought, ready to be lead by skillful teachers much further than is presently possible at the high schools of this country.

Hungary and Belgium also have long traditions of talent search and nurture from which Arnold and his helpers have profited.

I can testify about a similar experience in Germany, where my parents sent me to a new school founded in Hamburg, Germany, a few years after the first World War, by a group of teachers who believed in "unlimited horizons" of learning for boys and girls between 10 and 18. Though the science and math training at the Lichtwarck Schule was no different from the traditional fare of other Hamburgian schools, in all other subjects—languages, music, art and handcraft, history and government, art appreciation, and philosophy—we received far more stimulation and we were given a good many more opportunities of independent studies than were available to other Hamburgian high school students.

The blessings of early nurture of intellectual curiosity by the careful attention of excellent teachers make the recipients desirous of conferring similar blessings on the next generation.

When Arnold came to Chicago in 1922 he was ready to study; but he first went to an engineering school and only in 1925 he enrolled in the study of higher mathematics at the University of Chicago.

I know many other colleagues whose love of mathematics was kindled while they were studying engineering. Raoul Bott started out as an engineer-