Lecture Notes in Mathematics

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Lothar Göttsche

Hilbert Schemes of Zero-Dimensional Subschemes of Smooth Varieties

Springer-Verlag

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Springer-Verlag
Berlin Heidelberg New York
London Paris Tokyo
Hong Kong Barcelona
Budapest

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Mathematics Subject Classification (1991): 14C05, 14N10, 14D22

ISBN 3-540-57814-5 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-57814-5 Springer-Verlag New York Berlin Heidelberg

Printed in Germany

SPIN: 10078819 46/3140-543210 - Printed on acid-free paper

Introduction

Let X be a smooth projective variety over an algebraically closed field k. The easiest examples of zero-dimensional subschemes of X are the sets of n distinct points on X. These have of course length n, where the length of a zero-dimensional subscheme Z is $dim_k H^0(Z, \mathcal{O}_Z)$. On the other hand these points can also partially coincide and then the scheme structure becomes important. For instance subschemes of length 2 are either two distinct points or can be viewed as pairs (p,t), where p is a point of X and t is a tangent direction to X at p.

The main theme of this book is the study of the Hilbert scheme $X^{[n]} := \mathrm{Hilb}^n(X)$ of subschemes of length n of X; this is a projective scheme parametrizing zero-dimensional subschemes of length n on X. For $n = 1,2$ the Hilbert scheme $X^{[n]}$ is easy to describe; $X^{[1]}$ is just X itself and $X^{[2]}$ can be obtained by blowing up $X \times X$ along the diagonal and taking the quotient by the obvious involution, induced by exchanging factors in $X \times X$.

We will often be interested in the case where $X^{[n]}$ is smooth; this happens precisely if $n \leq 3$ or dim $X \leq 2$. If X is a curve, $X^{[n]}$ coincides with the n^{th} symmetric power of X, $X^{(n)}$; more generally, the natural set-theoretic map $X^{[n]} \to X^{(n)}$ associating to each subscheme its support (with multiplicities) gives a natural desingularization of $X^{(n)}$ whenever $X^{[n]}$ is smooth.

The case dim $X = 2$ is particularly important as this desingularization turns out to be crepant; that is, the canonical bundle on $X^{[n]}$ is the pullback of the dualizing sheaf on $X^{(n)}$ (in particular $X^{(n)}$ has Gorenstein singularities). In this case, $X^{[n]}$ is an interesting $2n$-dimensional smooth variety in its own right. For instance, Beauville [Beauville (1),(2),(3)] used the Hilbert scheme of a K3-surface to construct examples of higher-dimensional symplectic manifolds.

One of the main aims of the book is to understand the cohomology and Chow rings of Hilbert schemes of zero-dimensional subschemes. In chapter 2 we compute Betti numbers of Hilbert schemes and related varieties in a rather general context using the Weil conjectures; in chapter 3 and 4 the attention is focussed on easier and more special cases, in which one can also understand the ring structure of Chow and cohomology rings and give some enumerative applications.

In chapter 1 we recall some fundamental facts, that will be used in the rest of the book. First in section 1.1, we give the definition and the most important properties of $X^{[n]}$; then in section 1.2 we explain the Weil conjectures in the form in which we are later going to use them in order to compute Betti numbers of Hilbert schemes, and finally in section 1.3 we introduce the punctual Hilbert scheme, which parametrizes subschemes concentrated in a point of a smooth variety. We hope that the non-expert reader will find in particular sections 1.1 and 1.2 useful as a quick reference.

In chapter 2 we compute the Betti numbers of $S^{[n]}$ for S a surface, and of

KA_{n-1} for A an abelian surface, using the Weil conjectures. Here KA_{n-1} is a symplectic manifold, defined as the kernel of the map $A^{[n]} \to A$ given by composing the natural map $A^{[n]} \to A^{(n)}$ with the sum $A^{(n)} \to A$; it was introduced by Beauville [Beauville (1),(2),(3)].

We obtain quite simple power series expressions for the Betti numbers of all the $S^{[n]}$ in terms of the Betti numbers of S. Similar results hold for the KA_{n-1}. The formulas specialize to particularly simple expressions for the Euler numbers of $S^{[n]}$ and KA_{n-1}. It is noteworthy that the Euler numbers can also be identified as the coefficients in the q-development of certain modular functions and coincide with the predictions of the orbifold Euler number formula about the Euler numbers of crepant resolutions of orbifolds conjectured by the physicists. The formulas for the Betti numbers of the $S^{[n]}$ and KA_{n-1} lead to the conjecture of similar formulas for the Hodge numbers. These have in the meantime been proven in a joint work with Wolfgang Soergel [Göttsche-Soergel (1)]. One sees that also the signatures of $S^{[n]}$ and KA_{n-1} can be expressed in terms of the q-development of modular functions. The formulas for the Hodge numbers of $S^{[n]}$ have also recently been obtained independently by Cheah [Cheah (1)] using a different technique.

Computing the Betti numbers of $X^{[n]}$ can be viewed as a first step towards understanding the cohomology ring. A detailed knowledge of this ring or of the Chow ring of $X^{[n]}$ would be very useful, for instance in classical problems in enumerative geometry or in computing Donaldson polynomials for the surface X.

In section 2.5 various triangle varieties are introduced; by triangle variety we mean a variety parametrizing length 3 subschemes together with some additional structure. We then compute the Betti numbers of $X^{[3]}$ and of these triangle varieties for X smooth of arbitrary dimension, again by using the Weil conjectures.

The Weil conjectures are a powerful tool whose use is not as widely spread as it could be; we hope that the applications given in chapter 2 will convince the reader that they are not only important theoretically, but also quite useful in many concrete cases.

Chapters 3 and 4 are more classical in nature and approach then chapter 2. Chapter 3 uses Hilbert schemes of zero-dimensional subschemes to construct and study varieties of higher order data of subvarieties of smooth varieties. Varieties of higher order data are needed to give precise solutions to classical problems in enumerative algebraic geometry concerning contacts of families of subvarieties of projective space. The case that the subvarieties are curves has already been studied for a while in the literature [Roberts-Speiser (1),(2),(3)], [Collino (1)], [Colley-Kennedy (1)]. We will deal with subvarieties of arbitrary dimension and construct varieties of second and third order data. As a first application we compute formulas for the numbers of higher order contacts of a smooth projective variety with linear subvarieties in the ambient projective space. For a different and more general construction, which is however also more difficult to treat, as well as for examples of the type of problem that can be dealt with, we also refer the reader to [Arrondo-Sols-Speiser (1)].

The last chapter is the most elementary and classical of the book. We describe the Chow ring of the relative Hilbert scheme of three points of a $\mathbf{P}^2$ bundle. The main example one has in mind is the tautological $\mathbf{P}^2$-bundle over the Grassmannian of two-planes in $\mathbf{P}^n$. In this case it turns out hat our variety is a blow up of $(\mathbf{P}^n)^{[3]}$. This fact has been used in [Rosselló (2)] to determine the Chow ring of $(\mathbf{P}^3)^{[3]}$. The techniques we use are mostly elementary, for instance a study of the relative Hilbert scheme of finite length subschemes in a $\mathbf{P}^1$-bundle; I do however hope that the reader will find them useful in applications.

For a more detailed description of their contents the reader can consult the introductions of the chapters.

The various chapters are reasonably independent from each other; chapters 2, 3 and 4 are independent of each other, chapter 2 uses all of chapter 1, chapter 3 uses only the sections 1.1 and 1.3 of chapter 1 and chapter 4 uses only section 1.1.

To read this book the reader only needs to know the basics of algebraic geometry. For instance the knowledge of [Hartshorne (1)], is certainly enough, but also that of [Eisenbud-Harris (1)] suffices for reading most parts of the book. At some points a certain familiarity with the functor of points (like in the last chapter of [Eisenbud-Harris (1)]) will be useful. Of course we expect the reader to accept some results without proof, like the existence of the Hilbert scheme and obviously the Weil conjectures.

The book should therefore be of interest not only to experts but also to graduate students and researchers in algebraic geometry not familiar with Hilbert schemes of points.

Acknowledgements

I want to thank Professor Andrew Sommese, who has made me interested in Hilbert schemes of points. While I was still studying for my Diplom he proposed the problem on Betti numbers of Hilbert schemes of points on a surface, with which my work in this field has begun. He also suggested that I might try to use the Weil conjectures. After my Diplom I studied a year with him at Notre Dame University and had many interesting conversations. During most of the time in which I worked on the results of this book I was at the Max-Planck-Institut für Mathematik in Bonn. I am very grateful to Professor Hirzebruch for his interest and helpful remarks. For instance he has made me interested in the orbifold Euler number formula. Of course I am also very grateful for having had the possibility of working in the inspiring atmosphere of the Max-Planck-Institut.

I also want to thank Professor Iarrobino, who made me interested in the Hilbert function stratification of $\mathrm{Hilb}^n(k[[x,y]])$. Finally I am very thankful to Professor Ellingsrud, with whom I had several very inspiring conversations.

Contents

1. Fundamental facts

In this work we want to study the Hilbert scheme $X^{[n]}$ of subschemes of length n on a smooth variety. For this we have to review some concepts and results. In [Grothendieck (1)] the Hilbert scheme was defined and its existence proven. We repeat the definition in paragraph 1.1 and list some results about $X^{[n]}$. $X^{[n]}$ is related to the symmetric power $X^{(n)}$ via the Hilbert-Chow morphism $\omega_n : X^{[n]}_{red} \longrightarrow X^{(n)}$. We will use it to define a stratification of $X^{[n]}$. In chapter 2 we want to compute the Betti numbers of Hilbert schemes and varieties that can be constructed from them by counting their points over finite fields and applying the Weil conjectures. Therefore we give a review of the Weil conjectures in 1.2. Then we count the points of the symmetric powers $X^{(n)}$ of a variety X, because we will use this result in chapter 2. In 1.3 we study the punctual Hilbert scheme $\mathrm{Hilb}^n(k[[x_1, \dots, x_d]])$, parametrizing subschemes of length n of a smooth d-dimensional variety concentrated in a fixed point. In particular we give the stratification of Iarrobino by the Hilbert function of ideals.

1.1. The Hilbert scheme

Let T be a locally noetherian scheme, X a quasiprojective scheme over T and $\mathcal{L}$ a very ample invertible sheaf on X over T.

Definition 1.1.1. [Grothendieck (1)] Let $\mathcal{H}ilb(X/T)$ be the contravariant functor from the category $\underline{\mathrm{Schln}_T}$ of locally noetherian T-schemes to the category $\underline{\mathrm{Ens}}$ of sets, which for locally noetherian T-schemes U, V and a morphism $\phi : V \longrightarrow U$ is given by

$$\mathcal{H}ilb(X/T)(U) = \Big\{ Z \subset X \times_T U \text{ closed subscheme, flat over } U \Big\}$$
$$\mathcal{H}ilb(X/T)(\phi) : \mathcal{H}ilb(X/T)(U) \longrightarrow \mathcal{H}ilb(X/T)(V); Z \longmapsto Z \times_U V.$$

Let U be a locally noetherian T-scheme, $Z \subset X \times_T U$ a subscheme, flat over U. Let $p : Z \longrightarrow X$, $q : Z \longrightarrow U$ be the projections and $u \in U$. We put $Z_u = q^{-1}(u)$. The Hilbert polynomial of Z in u is

$$P_u(Z)(m) := \chi(\mathcal{O}_{Z_u}(m)) = \chi(\mathcal{O}_{Z_u} \otimes_{\mathcal{O}_Z} p^*(\mathcal{L}^m)).$$

$P_u(Z)(m)$ is a polynomial in m and independent of $u \in U$, if U is connected. For every polynomial $P \in \mathbf{Q}[x]$ let $\mathcal{H}ilb^P(X/T)$ be the subfunctor of $\mathcal{H}ilb(X/T)$ defined by

$$\mathcal{H}ilb^P(X/T)(U) = \left\{ \begin{array}{c} Z \subset X \times_T U \\ \text{closed subscheme} \end{array} \middle| \begin{array}{c} Z \text{ is flat over } U \text{ and} \\ P_u(Z) = P \text{ for all } u \in U \end{array} \right\}.$$

Theorem 1.1.2 [Grothendieck (1)]. *Let X be projective over T. Then for every polynomial $P \in \mathbf{Q}[x]$ the functor $\mathcal{H}ilb^P(X/T)$ is representable by a projective T-scheme* $\mathrm{Hilb}^P(X/T)$. *$\mathcal{H}ilb(X/T)$ is represented by*

$$\mathrm{Hilb}(X/T) := \bigcup_{P \in \mathbf{Q}[x]} \mathrm{Hilb}^P(X/T).$$

For an open subscheme $Y \subset X$ the functor $\mathcal{H}ilb^P(Y/T)$ is represented by an open subscheme

$$\mathrm{Hilb}^P(Y/T) \subset \mathrm{Hilb}^P(X/T).$$

Definition 1.1.3. $\mathrm{Hilb}(X/T)$ is the Hilbert scheme of X over T. If T is $spec(k)$ for a field k, we will write $\mathrm{Hilb}(X)$ instead of $\mathrm{Hilb}(X/T)$ and $\mathrm{Hilb}^P(X)$ instead of $\mathrm{Hilb}^P(X/T)$. If P is the constant polynomial $P = n$, then $\mathrm{Hilb}^n(X/T)$ is the relative Hilbert scheme of subschemes of length n on X over T. If T is the spectrum of a field, we will write $X^{[n]}$ for $\mathrm{Hilb}^n(X) = \mathrm{Hilb}^n(X/spec(k))$. $X^{[n]}$ is the Hilbert scheme of subschemes of length n on X.

If U is a locally noetherian T-scheme, then $\mathcal{H}ilb^n(X/T)(U)$ is the set

$$\left\{ \text{closed subschemes } Z \subset X \times_T U \;\middle|\; Z \text{ is flat of degree } n \text{ over } U \right\}.$$

In particular we can identify the set $X^{[n]}(k)$ of k-valued points of $X^{[n]}$ with the set of closed zero-dimensional subschemes of length n of X which are defined over k. In the simplest case such a subscheme is just a set of n distinct points of X with the reduced induced structure. The length of a zero-dimensional subscheme $Z \subset X$ is $dim_k H^0(Z, \mathcal{O}_Z)$. The fact that $\mathrm{Hilb}^n(X/T)$ represents the functor $\mathcal{H}ilb^n(X/T)$ means that there is a universal subscheme

$$Z_n(X/T) \subset X \times_T \mathrm{Hilb}^n(X/T),$$

which is flat of degree n over $\mathrm{Hilb}^n(X/T)$ and fulfills the following universal property: for every locally noetherian T-scheme U and every subscheme $Z \subset X \times_T U$ which is flat of degree n over U there is a unique morphism

$$f_Z : U \longrightarrow \mathrm{Hilb}^n(X/T)$$

such that

$$Z = (1_X \times_T f_Z)^{-1}(Z_n(X/T)).$$

For $T = spec(k)$ we will again write $Z_n(X)$ instead of $Z_n(X/T)$.

Remark 1.1.4. It is easy to see from the definitions that $Z_n(X/T)$ represents the functor $\mathcal{Z}_n(X/T)$ from the category of locally noetherian schemes to the category of sets which is given by

$$\mathcal{Z}_n(X/T)(U)\left\{(Z,\sigma)\ \middle|\ \begin{array}{c} Z \text{ closed subschemes of } X\times_T U, \\ \text{flat of degree } n \text{ over } U, \\ \sigma: U \longrightarrow Z \text{ a section of the projection } Z \longrightarrow U \end{array}\right\},$$

$$\mathcal{Z}_n(X/T)(\Phi): \mathcal{Z}_n(X/T)(U) \longrightarrow \mathcal{Z}_n(X/T)(V);$$
$$(Z,\sigma) \longmapsto (Z\times_U V, \sigma\circ\Phi)$$

(U, V locally noetherian schemes $\Phi: V \longrightarrow U$).

For the rest of section 1.1 let X be a smooth projective variety over the field k.

Definition 1.1.5. Let $G(n)$ be the symmetric group in n letters acting on X^n by permuting the factors. The geometric quotient $X^{(n)} := X^n/G(n)$ exists and is called the n-fold symmetric power of X. Let

$$\Phi_n : X^n \longrightarrow X^{(n)}$$

be the quotient map.

$X^{(n)}$ parametrizes effective zero-cycles of degree n on X, i.e. formal linear combinations $\sum n_i[x_i]$ of points x_i in X with coefficients $n_i \in \mathbb{N}$ fulfilling $\sum n_i = n$. $X^{(n)}$ has a natural stratification into locally closed subschemes:

Definition 1.1.6. Let $\nu = (n_1, \ldots, n_r)$ be a partition of n. Let

$$\Delta_{n_i} := \left\{(x_1,\ldots,x_{n_i})\ \middle|\ x_1 = x_2 = \ldots = x_{n_i}\right\} \subset X^{n_i}$$

be the diagonal and

$$X^n_\nu := \prod_{i=1}^r \Delta_{n_i} \subset \prod_{i=1}^r X^{n_i} = X^n.$$

Then we set

$$\overline{X^{(n)}_\nu} := \Phi_n(X^n_\nu)$$

and

$$X^{(n)}_\nu := \overline{X^{(n)}_\nu} \setminus \bigcup_{\mu>\nu} \overline{X^{(n)}_\mu}.$$

Here $\mu > \nu$ means that μ is a coarser partition then ν.

The geometric points of $X_\nu^{(n)}$ are

$$X_\nu^{(n)}(\bar{k}) = \left\{ \sum n_i[x_i] \in X^{(n)}(\bar{k}) \;\middle|\; \text{the points } x_i \text{ are pairwise distinct} \right\}.$$

The $X_\nu^{(n)}$ form a stratification of $X^{(n)}$ into locally closed subschemes, i.e they are locally closed subschemes, and every point of $X^{(n)}$ lies in a unique $X_\nu^{(n)}$. The relation between $X^{[n]}$ and $X^{(n)}$ is given by:

Theorem 1.1.7 [Mumford-Fogarty (1) 5.4]. *There is a canonical morphism (the Hilbert Chow morphism)*

$$\omega_n : X_{red}^{[n]} \longrightarrow X^{(n)},$$

which as a map of points is given by

$$Z \mapsto \sum_{x \in X} len(Z_x)[x].$$

So the above stratification of $X^{(n)}$ induces a stratification of $X_{red}^{[n]}$:

Definition 1.1.8. For every partition ν of n let

$$X_\nu^{[n]} := \omega_n^{-1}(X_\nu^{(n)}).$$

Then the $X_\nu^{[n]}$ form a stratification of $X_{red}^{[n]}$ into locally closed subschemes.

For $\nu = (n_1, \dots, n_r)$ the geometric points of $X_\nu^{[n]}$ are just the unions of subschemes $Z_1, \dots, Z_r$, where each Z_i is a subscheme of length n_i of X concentrated in a point x_i and the x_i are distinct.

1.2. The Weil conjectures

We will use the Weil conjectures to compute the Betti numbers of Hilbert schemes. They have been used before to compute Betti numbers of algebraic varieties, at least since in [Harder-Narasimhan (1)] they were applied for moduli spaces of vector bundles on smooth curves.

Let X be a projective scheme over a finite field $\mathbb{F}_q$, let $\overline{\mathbb{F}}_q$ be an algebraic closure of $\mathbb{F}_q$ and $\overline{X} := X \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$. The *geometric Frobenius*

$$F_X : X \longrightarrow X$$

is the morphism of X to itself which as a map of points is the identity and the map $a \mapsto a^q$ on the structure sheaf $\mathcal{O}_X$. The geometric Frobenius of $\overline{X}$ over $\mathbb{F}_q$ is

$$F_q := F_X \times 1_{\overline{\mathbb{F}}_q}.$$

The action of F_q on the geometric points $X(\overline{\mathbb{F}}_q)$ is the inverse of the action of the Frobenius of $\mathbb{F}_q$. As this is a topological generator of the Galois group $\mathrm{Gal}(\overline{\mathbb{F}}_q, \mathbb{F}_q)$, a point $x \in X(\overline{\mathbb{F}}_q)$ is defined over $\mathbb{F}_q$, if and only if $x = F_q(x)$. For a prime l which does not divide q let $H^i(\overline{X}, \mathbf{Q}_l)$ be the i^{th} l-adic cohomology group of $\overline{X}$ and

$$\begin{aligned} b_i(\overline{X}) &:= dim_{\mathbf{Q}_l}(H^i(\overline{X}, \mathbf{Q}_l)), \\ p(\overline{X}, z) &:= \sum b_i(\overline{X}) z^i, \\ e(\overline{X}) &:= \sum (-1)^i b_i(\overline{X}). \end{aligned}$$

$b_i(\overline{X})$ is independent of l. We will denote the action of F_q^* on $H^r(\overline{X}, \mathbf{Q}_l)$ by $F_q^*|_{H^r(\overline{X}, \mathbf{Q}_l)}$. The *zeta-function* of X over $\mathbb{F}_q$ is the power series

$$Z_q(X,t) := \exp\left(\sum_{n>0} |X(\mathbb{F}_{q^n})| \frac{t^n}{n}\right).$$

Here $|M|$ denotes the number of elements in a finite set M.

Let X be a smooth projective variety over the complex numbers $\mathbf{C}$. Then X is already defined over a finitely generated extension ring R of $\mathbb{Z}$, i.e. there is a variety X_R defined over R such that $X_R \times_R \mathbf{C} = X$. For every prime ideal p of R let $X_p := X_R \times_R R/p$. There is a nonempty open subset $U \subset spec(R)$ such that X_p is smooth for all $p \in U$, and the l-adic Betti-numbers of X_p coincide with those of X for all primes l different from the characteristic of A/p (cf. [Kirwan (1) 15.], [Bialynicki-Birula, Sommese (1) 2.]. If $m \subset R$ is a maximal ideal lying in U for which R/m is a finite field $\mathbb{F}_q$ of characteristic $p \neq l$, we call X_m a good reduction of X modulo q.

Theorem 1.2.1. *(Weil conjectures* [Deligne (1)], *cf.* [Milne (1)], [Mazur (1)]*)*

(1) *$Z_q(X,t)$ is a rational function*

$$Z_q(X,t) = \prod_{r=0}^{2d} Q_r(X,t)^{(-1)^{r+1}}$$

with $Q_r(X,t) = \det(1 - tF_q^*|_{H^r(\overline{X},\mathbf{Q}_l)})$.

(2) $Q_r(X,t) \in \mathbb{Z}[t]$.

(3) *The eigenvalues $\alpha_{i,r}$ of $F_q^*|_{H^r(\overline{X},\mathbf{Q}_l)}$ have the absolute value $|\alpha_{ir}| = q^{r/2}$ with respect to any embedding into the complex numbers.*

(4)
$$Z_q(X, 1/q^d t) = \pm q^{e(\overline{X})/2} t^{e(\overline{X})} Z_q(X,t).$$

(5) *If X is a good reduction of a smooth projective variety Y over $\mathbf{C}$, then we have*

$$b_i(Y) = b_i(\overline{X}) = \deg(Q_i(\overline{X},t)).$$

Remark 1.2.2. Let $F(t, s_1, \ldots, s_m) \in \mathbf{Q}[t, s_1, \ldots, s_m]$ be a polynomial. Let X and S be smooth projective varieties over $\mathbb{F}_q$ such that

$$|X(\mathbb{F}_{q^n})| = F(q^n, |S(\mathbb{F}_{q^n})|, \ldots, |S(\mathbb{F}_{q^{nm}})|)$$

holds for all $n \in \mathbb{N}$. Then we have

$$p(\overline{X}, -z) = F(z^2, p(\overline{S}, -z), \ldots, p(\overline{S}, -z^m)).$$

If X and S are good reductions of smooth varieties Y and U over $\mathbf{C}$, we have:

$$p(Y, -z) = F(z^2, p(U, -z), \ldots, p(U, -z^m)).$$

Proof: Let $\alpha_1, \ldots, \alpha_s$ be pairwise distinct complex numbers and $h_1, \ldots, h_s \in \mathbf{Q}$. We put

$$Z((\alpha_i, h_i)_i) := \exp\left(\sum_{n>0}\left(\sum_{i=1}^{s} h_i \alpha_i^n\right)\frac{t^n}{n}\right).$$

Then we have

$$Z((\alpha_i, h_i)_i) = \prod_{i=1}^{s}(1 - \alpha_i)^{-h_i}.$$

So we can read off the set of pairs $\{(\alpha_1, h_1), \dots (\alpha_s, h_s)\}$ from the function $Z((\alpha_i, h_i)_i)$. For each $c \in \mathbf{C}$ let $r(c) := 2\log_q(|c|)$. By theorem 1.2.1 we have: for a smooth projective variety W over $\mathbb{F}_q$ there are distinct complex numbers $(\beta_i)_{i=1}^t \in \mathbf{C}$ and integers $(l_i)_{i=1}^t \in \mathbb{Z}$ such that

$$|W(\mathbb{F}_{q^n})| = \sum_{i=1}^t l_i \beta_i^n$$

for all $n \in \mathbb{N}$. Furthermore we have $r(\beta_i) \in \mathbb{Z}_{\geq 0}$ and

$$(-1)^k b_k(W) = \sum_{r(\beta_i)=k} l_i$$

for all $k \in \mathbb{Z}_{\geq 0}$. Let $\beta_1, \dots, \beta_t \in \mathbf{C}$, $l_1, \dots, l_t \in \mathbb{Z}$ be the corresponding numbers for S. Then we have for all $n \in \mathbb{N}$:

$$|X(\mathbb{F}_{q^n})| = F\left(q^n; \sum_{i=1}^t l_i \beta_i^n, \dots, \sum_{i=1}^t l_i \beta_i^{mn}\right).$$

Let $\delta_1, \dots, \delta_r$ be the distinct complex numbers which appear as monomials in q and the γ_i in

$$F\left(q, \sum_{i=1}^t l_i \beta_i, \dots, \sum_{i=1}^t l_i \beta_i^m\right).$$

Then there are rational numbers $n_1, \dots, n_r$ such that

$$|X(\mathbb{F}_{q^n})| = \sum_{i=1}^r n_i \delta_i^n$$

for all $n \in \mathbb{N}$ and

$$(-1)^k b_k(\overline{X}) = \sum_{r(\delta_j)=k} n_j$$

for all $k \in \mathbb{Z}_{\geq 0}$. We see from the definitions that $\sum_{r(\delta_j)=k} n_j$ is the coefficient of z^k in $F(z^2, p(S, -z), \dots, p(S, -z^m))$. □

We finish by showing how to compute the number of points of the symmetric power $X^{(n)}$ for a variety X over $\mathbb{F}_q$. The geometric Frobenius $F := F_q$ acts on $X^{(n)}(\overline{\mathbb{F}}_q)$ by

$$F\left(\sum n_i[x_i]\right) = \sum n_i[F(x_i)],$$

and $X^{(n)}(\mathbb{F}_q)$ is the set of effective zero-cycles of degree n on X which are invariant under the action of F.

Definition 1.2.3. A zero-cycle of the form

$$\sum_{i=0}^{r}[F^i(x)] \qquad \text{with } x \in X(\mathbb{F}_{q^r}) \setminus \bigcup_{j|r} X(\mathbb{F}_{q^j})$$

is called a *primitive zero-cycle* of degree r on X over $\mathbb{F}_q$. The set of primitive zero-cycles of degree r on X over $\mathbb{F}_q$ will be denoted by $P_r(X, \mathbb{F}_q)$.

Remark 1.2.4.

(1) Each element $\xi \in X^{(n)}(\mathbb{F}_q)$ has a unique representation as a linear combination of distinct primitive zero-cycles over $\mathbb{F}_q$ with positive integer coefficients.

(2) $|X(\mathbb{F}_{q^n})| = \sum_{r|n} r \cdot |P_r(X, \mathbb{F}_q)|$

(3) $Z_q(X,t) = \sum_{n \geq 0} |X^{(n)}(\mathbb{F}_q)| t^n$,

i.e. $Z_q(X,t)$ is the generating function for the numbers of effective zero-cycles of X over $\mathbb{F}_q$.

Proof: (1) Let $\xi = \sum_{i=1}^{r} n_i[x_i] \in X^{(n)}(\mathbb{F}_q)$, where $x_1, \ldots, x_r$ are distinct elements of $X(\overline{\mathbb{F}}_q)$. For all j let $\xi_j := \sum_{n_i \geq j}[x_i] \in X^{(n)}(\mathbb{F}_q)$. Then we have $\xi = \sum_j \xi_j$, and it suffices to prove the result for the ξ_j. So we can assume that ξ is of the form $\xi = \sum_{i=1}^{r}[x_i]$ with pairwise distinct $x_i \in X(\overline{\mathbb{F}}_q)$. As we have $F(\xi) = \xi$, there is a permutation σ of $\{1, \ldots, r\}$ with $F(x_i) = x_{\sigma(i)}$ for all i. Let $M_1, \ldots, M_s \subset \{1, \ldots, r\}$ be the distinct orbits under the action of σ. Then we set

$$\eta_j := \sum_{i \in M_j} [x_i]$$

for $j = 1, \ldots s$. Then $\xi = \sum_{j=1}^{s} \eta_j$ is the unique representation of ξ as a sum of primitive zero-cycles.

(2) follows immediately from the definitions. From (1) we have

$$\begin{aligned}\sum_{n \geq 0} |X^{(n)}(\mathbb{F}_q)| t^n &= \prod_{r \geq 1} (1 - t^r)^{-|P_r(X, \mathbb{F}_q)|} \\ &= \exp\left(\sum_{r \geq 1} |P_r(X, \mathbb{F}_q)| \left(\sum_{m \geq 1} \frac{t^{rm}}{m}\right)\right) \\ &= \exp\left(\sum_{n=0}^{\infty} \left(\sum_{r|n} r \cdot |P_r(X, \mathbb{F}_q)|\right) \frac{t^n}{n}\right) \\ &= Z_q(X,t).\end{aligned}$$

So (3) holds. □

1.3. The punctual Hilbert scheme

Let $R := k[[x_1, \dots, x_d]]$ be the field of formal power series in d variables over a field k. Let $\mathbf{m} = (x_1, \dots, x_d)$ be the maximal ideal of R.

Definition 1.3.1. Let $I \subset R$ be an ideal of colength n. The *Hilbert function* $T(I)$ of I is the sequence $T(I) = (t_i(I))_{i \geq 0}$ of non-negative integers given by

$$t_i = dim_k(\mathbf{m}^i/(I \cap \mathbf{m}^i + \mathbf{m}^{i+1})).$$

If $T = (t_i)_{i \geq 0}$ is a sequence of non-negative integers, of which only finitely many do not vanish, we put $|T| = \sum t_i$. The *initial degree* d_0 of T is the smallest i such that $t_i < \binom{d+i-1}{i}$.

Let $R_i := \mathbf{m}^i/\mathbf{m}^{i+1}$ and $I_i := (\mathbf{m}^i \cap I)/(\mathbf{m}^{i+1} \cap I)$. Then R_i is the space of forms of degree i in R and I_i the space of initial forms of I (i.e. the forms of minimal degree among elements of I) of degree i, and we have:

$$t_i(I) = dim_k(R_i/I_i).$$

Let $I \subset R$ be an ideal of colength n and $T = (t_i)_{i \geq 0}$ the Hilbert function of I.

Lemma 1.3.2.

(1)
$$dim(\mathbf{m}^j/I \cap \mathbf{m}^j) = \sum_{i \geq j} t_i$$

holds for all $j \geq 0$. In particular we have $|T| = n$.

(2) $I \supset \mathbf{m}^n$.

Proof: Let $Z := R/I$, and Z_i the image of $\mathbf{m}^i$ under the projection $R \longrightarrow Z$. Then we have

$$\bigcap_{i \geq 0} Z_i = 0.$$

As Z is finite dimensional, there exists an i_0 with $Z_{i_0} = 0$. For such an i_0 we have $I \supset \mathbf{m}^{i_0}$. There is an isomorphism

$$Z_j = \mathbf{m}^j/(\mathbf{m}^j \cap I) \cong \oplus_{i=j}^{i_0-1} R_i/I_i$$

of k-vector spaces, and $R_i/I_i = 0$ holds for $i \geq i_0$. If we choose i_0 to be minimal, then $R_i/I_i \neq 0$ holds for $i < i_0$. So we get (1). If $t_j = 0$ for some j, then $I \supset \mathbf{m}^j$. Thus (2) follows from $|T| = n$. □