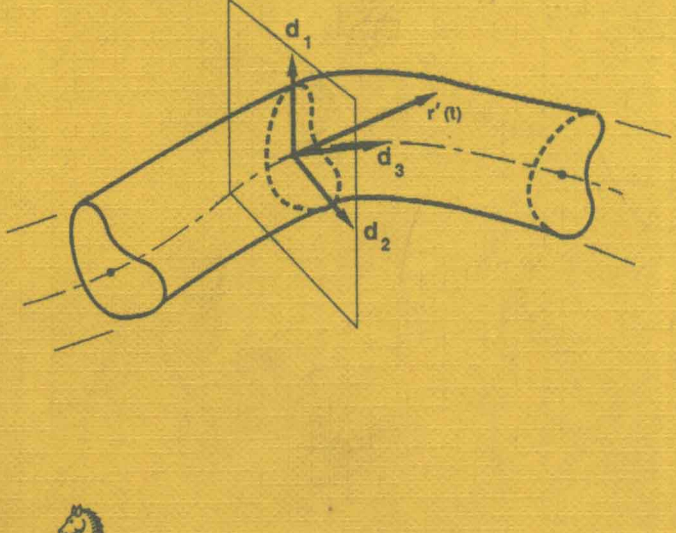
Lecture Notes in Mathematics

1489

Alexander Mielke

Hamiltonian and Lagrangian Flows on Center Manifolds

with Applications to Elliptic Variational Problems



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Für Bärbel, Annemarie, Lisabet und Frieder

Preface

It is the aim of this work to establish connections between three fields which seem only loosely related from the usual point of view. These fields are described by the following terms: Hamiltonian and Lagrangian systems, center manifold reduction, and elliptic variational problems. All three topics have had a period of fast development within the last two decades; and the interrelations have grown considerably. Here we want to consider just one facet at the intersection of all three fields, namely the implications of center manifold theory to the study of variational problems. The main tool for the analysis is the Hamiltonian point of view.

The original motivation for this work derives from my interest in Saint–Venant's problem. Having in mind the center manifold approach and hearing about the Galerkin or projection method to derive rod models, I thought it worthwile to study the connections between these two reduction procedures. However, it soon turned out that the tools to be developed involved fairly general ideas in Hamiltonian and Lagrangian system theory. I realized that many of the necessary results are known but spread over many sources or hidden in very abstract clothing. Often, the special needs for our objective were not directly covered. Thus, the plan evolved to write the abstract Part I on Hamiltonian and Lagrangian systems as self–contained as possible.

I made a controversial decision concerning the use of methods and notations from differential geometry. Since many applied researchers are not familiar with differential forms and coordinate free analysis on manifolds, I avoided these tools as much as possible. On the one hand the center manifold is a local object and can be described by one coordinate chart; but on the other hand it is nevertheless a manifold and in order to define a Hamiltonian system on it, the use of differential forms is absolutely necessary. Additionally, the concept of Lie groups involves global manifolds. In this conflict I was guided by the idea to enable nonspecialists (I am one of them) to follow the analysis, give them a first contact to methods in analysis on manifolds and symplectic geometry, and, finally, to motivate them to study these methods in their own right. The reader should judge how far this goal is reached.

To appreciate the abstract methods of Part I it is strongly recommended to get involved in some of the applications studied in Part II (the easiest one is given in Section 8.2). For classical Hamiltonian systems the center manifold does not play an important role, since most systems are oscillatory, which leads to high-dimensional center manifolds. However, center manifolds in elliptic problems in cylinders, first found by Kirchgässner

[Ki82], reduce an infinite–dimensional system to a finite–dimensional one. In nonlinear elasticity one application is the Saint–Venant problem concerning the static deformations of a very long prismatic body. In mechanics this situation is modelled by rod theory which is an ordinary differential equation replacing the equilibrium equations of three–dimensional nonlinear elasticity. Such rod models can now be justified by the center manifold approach; and it was the question of finding the variational structure of these rod equations which brought my attention to this exciting field.

The research reported here was initiated during a one–year stay at Cornell University, where I had numerous, very stimulating discussions with Phil Holmes. I am very grateful to him. Additionally, I would like to thank all the persons who supplied me with interesting hints and helpful comments during the development of this work. Especially, I want to mention M. Beyer, T. Healey, H. Hofer, G. Iooss, J. Marsden, J. Moser, P. Slodowy, and E. Zehnder. Finally, my thanks extend to my former advisor and teacher K. Kirchgässner, who played a major role in my education in mathematics and their applications and who encouraged me whenever needed.

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Stuttgart, May 1991 *Alexander Mielke*

Contents

Chapter 1

Introduction

In applications, very often complex mathematical models are developed to describe certain phenomena in nature. However, in some circumstances it turns out that only a few critical modes are relevant for the basic effects. This is especially the case when systems with an equilibrium state near the threshold of instability are considered. Then it is desirable to have a method to reduce the complex system to a simpler one which only takes into account the amplitudes of the critical modes. The basic requirement for such a reduction procedure is that the reduced model is a faithful representation of the original problem, at least locally. This means that the solution sets of both systems should be in a one-to-one correspondence for the solutions one is interested in.

For steady problems such a method is provided by the Lyapunov-Schmidt reduction, whereas for time-dependent systems the center manifold reduction is available. There are further reduction methods which may not (yet) be justified in a mathematically rigorous way (in the sense of faithfulness), but are still widespread in the applied sciences due to their simplicity and their important results. Examples are the so-called Galerkin approximations [GH83, pg.417] and amplitude or modulation equations of Ginzburg-Landau type [NW69, DES71]. For the latter, initial results for a mathematical justification are given in [CE90, vH90, IM91, Mi91b].

For Hamiltonian and variational methods it is well-known that Galerkin approximations are best done on the Hamiltonian function and on the functional rather than on the associated differential equation. Then, the reduced problem maintains the Hamiltonian or variational structure; see e.g. [Ho86, BB87] for examples in fluid dynamics and [An72, Ow87] for elasticity theory. We will call these methods simply *projection methods*. It is our aim to recover these methods from a mathematical rigorous basis.

The main motivation for the present work is the study of elliptic variational problems on cylindrical domains, in particular Saint-Venant's problem for the deformations of long prismatic bodies (Ch. 11). It was first shown in [Ki82], that elliptic systems in cylinders can be considered as (ill-posed) evolutionary problems with the axial variable as time, and that these problems are accessible by the center manifold reduction. With this tool one is able to construct all solutions on the infinite cylinder which stay close to a given solution being independent of the axial variable. This method was extensively developed in [Fi84, Mi86a, Mi88b, Mi90, IV91]. Here we are not concerned with the technicalities necessary for proving the existence of center manifolds in this context.

Our interest lies in the question, what additional information about the reduced problem on the center manifold can be gained when the original elliptic problem was obtained as Euler–Lagrange equation of a variational problem. The best we can expect is that the reduced problem can be again understood as a reduced variational problem. For example, this is the case for the Lyapunov–Schmidt reduction, see Section 6.1. But for center manifold reduction, the desired result is, in general, false. Fortunately, in most applications it can be shown that the center manifold reduction of a variational problem is again variational. We state the conjecture that for *strongly elliptic* systems this is always true (Section 6.3).

The main idea to approach variational problems in cylinders is to exploit the distinguished role of the axial variable. Starting with the energy density $f = f(y, u, \nabla_y u, \dot{u})$, where $(y, t) \in \Sigma \times I\!R$ with Σ being the cross-section and $\dot{u} = \partial u / \partial t$, we define a Lagrangian

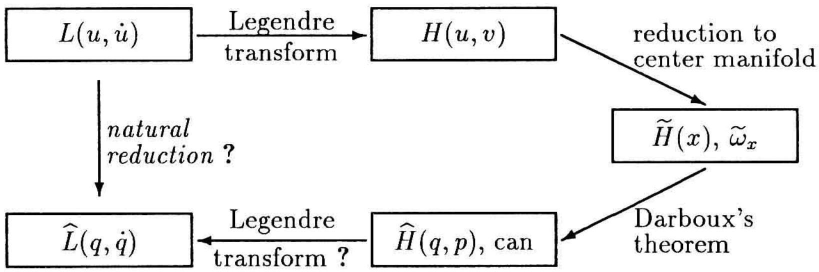
$$L(u, \dot{u}) = \int_\Sigma f(y, u, \nabla_y u, \dot{u})\, dy$$

where $(u, \dot{u})$ is now assumed to be an element of some function space over the cross-section Σ. Thus, the system can now be considered as a Lagrangian problem in an infinite-dimensional space. In this sense we use the words 'Lagrangian problem' and 'variational problem' synonymously. Associated to this Lagrangian formulation is a Hamiltonian formulation $H(u, v)$ which is be obtained by the Legendre transform $v = \partial L / \partial \dot{u}$. This relation can always be locally inverted due to the ellipticity of the problem.

Now, all the tools of Hamiltonian systems are at our disposal. For instance, bifurcations of solutions, which are periodic in the axial direction, have to obey the more stringend rules of Hamiltonian bifurcation theory. In [BM90] this idea is applied to the bifurcations from the Stokes family in the theory of surface waves.

Center manifolds are mostly studied in cases where they are stable. This explains why they are not used very often in Hamiltonian theory, since there the symmetry of the spectrum does not allow for a stable center manifold. However, the basis of the present work is the observation that the flow on the center manifold of a Hamiltonian system is again described by a reduced Hamiltonian system. Although this fact seems to be well-known in the realm of Hamiltonian theory, one hardly finds references to this; see e.g. [Kl82, Ch.3.1] for the linear case and [Po80, Ch.2E] for the case involving simple eigenvalues only. The only general treatment, the author is aware of, is given in [Mo77].

After recovering canonical coordinates on the center manifold we have to check whether it is possible to do the inverse Legendre transform; then a reduced variational problem is found. We represent this method in the following diagram.



This method constitutes the core of the work presented here. Of course, it can be considered independently of elliptic problems, purely on the level of Lagrangian and Hamiltonian systems. The abstract derivation of the method is done in Part I, and applications to several problems in continuum mechanics are given in Part II.

One remaining open question is to clarify the relations between the full Lagrangian L and the reduced Lagrangian $\widehat{L}$. The most natural relation would be that $\widehat{L}$ equals the restriction of L to the center manifold. If this holds, we use the notion of *natural reduction*. Note that this is exactly the case when the projection (or Galerkin) method yields the same reduced problem as our method. However, the results in this area (Section 6.5) are only preliminary and further research is needed.

On the contents

In Chapter 2 we introduce the necessary notations for Hamiltonian systems on manifolds and then give a basic introduction to the ideas of center manifold theory. Next we treat linear Hamiltonian systems and present some results on linear normal form theory, which are needed later on.

In Chapter 4 we consider center manifolds in Hamiltonian systems. In the first section we give the basic reduction theorem due to Moser [Mo77] and certain related results. In Section 4.2 we generalize the result to Poisson systems as follows. Let X be a Banach space which splits into $X_1 \times X_2$ where X_1 is finite-dimensional. Consider a Hamiltonian system on X given by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} J_1(x) & J_2(x) \\ J_3(x) & J_4(x) \end{pmatrix} \begin{pmatrix} \mathbf{d}_{x_1} H(x) \\ \mathbf{d}_{x_2} H(x) \end{pmatrix}. \tag{1.1}$$

Here, H is the Hamiltonian and $J = \begin{pmatrix} J_1 & J_2 \\ J_3 & J_4 \end{pmatrix} : X^* \to X$ defines the Poisson bracket $\{F, G\} = \mathbf{d}F(J\mathbf{d}G)$. We assume that (1.1) has a center manifold M_C being tangential to X_1 in the point $x_1 = 0$, i.e. $M_C = \{\,(x_1, h(x_1)) \in X \;:\; x_1 \in U_1 \subset X_1\,\}$, where the reduction function h satisfies $h(0) = Dh(0) = 0$.

Under the sole assumption that $J_4(0) : X_2^* \to X_2$ has a bounded inverse we show that the flow on M_C is described by

$$\dot{x}_1 = \widetilde{J}(x_1)\mathbf{d}_{x_1}\widetilde{H}(x_1). \tag{1.2}$$

The reduced Hamiltonian $\widetilde{H}$ is just the restriction of H onto M_C: $\widetilde{H}(x_1) = H(x_1, h(x_1))$. The operator $\widetilde{J}(x_1) : X_1^* \to X_1$ defines a reduced Poisson bracket on M_C and is given by

$$\widetilde{J}(x_1) = J_1 + (-J_1 B^* + J_2)(J_4 - BJ_2 - J_3 B^* + BJ_1 B^*)^{-1}(BJ_1 - J_3) \tag{1.3}$$

where $J_i = J_i(x_1, h(x_1))$ and $B = Dh(x_1)$, with B^* being the adjoint. In fact, this reduction method is not restricted to center manifolds but holds for every invariant submanifold with invertible J_4.

In Section 4.3 we treat the method of *flattening of center manifolds* in canonical coordinates. This procedure gives an effective way of calculating the reduced Hamiltonian in canonical coordinates, up to any given order. This method is especially useful for implementation, either numerically or by symbolic manipulations. Finally, we are concerned with the analyticity problem for center manifolds in Hamiltonian systems.

In Chapter 5 systems are studied which are invariant under the action of a Lie group. A short introduction to Lie groups is given in Section 5.1 and the classical *reduction of Hamiltonian system by symmetry* ([Ma81, MR86]) is outlined. Note that the reduction onto a center manifold is completely different from the classical reduction. There a reduced phase space is obtained by factoring with respect to a symmetry action and by restricting onto level surfaces of the corresponding first integrals. The center manifold, however, is characterized only by the dynamical behavior of the solutions lying on it. In particular, in Section 5.2 we show that systems, being invariant under the action of a Lie group, give rise to a reduced system on the center manifold which inherits all the symmetries of the original system. For this purpose we use a slice theorem to decouple the symmetry action. Furthermore we develop special versions of Poincaré's Lemma and Darboux's Theorem for the symmetry case. It follows then that the symmetry reduction can be done after the center manifold reduction.

Variational problems are treated in Chapter 6. First we consider the Lyapunov–Schmidt reduction and the projection method which both have the property that their reduced problem has a natural variational structure. We point out the essential differences to our approach and to get ideas what the desired results for Lagrangian systems are. As a byproduct, we obtain that the center manifold reduction in a gradient system is again a gradient system, but with respect to a non–flat metric.

Then we study the abstract setting of Lagrangian problems and their relation to canonical Hamiltonian systems. The general reduction method proposed above always leads from the original Lagrangian problem, via the Legendre transform and the center manifold reduction, to a reduced Hamiltonian system. Darboux's theorem then supplies canonical coordinates and we would like to transform back into a Lagangian problem. Yet, in general this is not possible, even when all possible canonical coordinates are taken into account. The following condition is necessary and sufficient for the existence of an associated Lagrangian problem. Denoting the linearization of the vector field in (1.2) at $x_1 = 0$ by $K_1 x_1$ this condition is simply given by

$$\dim(\operatorname{kernel} K_1) \leq \tfrac{1}{2} \dim X_1. \tag{1.4}$$

This result is very useful, since it makes the sometimes cumbersome calculation of the quadratic part of $\widetilde{H}$ superfluous. Using this condition the reduction procedure can be completed as given in the above diagram. The important question of natural reduction is discussed in Section 6.5. However, by now only a relaxed version of it is well understood.

Moreover, we consider symmetric Lagrangian systems which may have a *relative equilibrium* with respect to the action of a Lie group. Using augmented Hamiltonians we are able to construct a center manifold close to the relative equilibrium, such that the reduced Lagrangian system is again invariant under the reduced action. This theory will be basic for the understanding of Saint–Venant's problem.

In Chapter 7 we are concerned with nonautonomous Hamiltonian or Lagrangian systems. Under suitable uniformity conditions on the time–dependence, the existence of a time–dependent center manifold is known. Then the reduced symplectic structure depends on time also. Using an especially adapted version of Darboux's theorem we find canonical coordinates while keeping the time variable unchanged and preserving the qualitative time–dependence like (quasi–) periodicity. Moreover, for an autonomous system under a nonautonomous perturbation on a finite time–interval we show that the reduced system on the center manifold can be brought into the same form. This leads to applications in elliptic problems with localized perturbations, e.g. fluid flow over a bump.

Part II is dedicated to applications of the above theory to elliptic variational problems in cylindrical domains. In Chapter 8 we show that elliptic variational problems in cylinders admit a natural Lagrangian formulation, where the Lagrange function is obtained by simply integrating the density over the cross–section rather than over the whole cylinder. Moreover, the ellipticity condition shows that this Lagrangian problem can always be converted into an associated Hamiltonian problem by the generalized Legendre transformation. Now, the reduction onto a center manifold yields a reduced Hamiltonian system. The question is whether the reduced system on the center manifold again can be understood as the Euler–Lagrange equations of a reduced variational problem. But this can be answered affirmatively if condition (1.4) holds. This provides the first rigorous reduction procedure for elliptic variational problems in cylindrical domains to a finite–dimensional variational problem.

As applications we treat several elliptic problems with increasing complexity. Most of them were already discussed before, however the Hamiltonian structure was not exploited then. The first is just an introductory example to show the basic ideas without involving too much technicalities (Section 8.2). Then we study a nonautonomous problem, describing steady internal waves in a channel in the presence of a small localized perturbation (bump) at the bottom of the channel. The previous results in [Mi86b, Mi88b] can now be obtained much more easily and more completely.

In Chapter 9 steady surface waves under the influence of capillarity and gravity are investigated. In particular, we discuss the question of solitary waves for small Bond number and show that no multi–solitons can bifurcate at Froude number equal to 1.

The fourth example comes from elasticity and considers the deformation of a two–dimensional strip under tensile loading. For certain materials there exists a critical load under which the strip starts to deviate from the homogeneous deformation and exhibits localized necks. We establish the corresponding elliptic variational problem and find the associated reduced variational problem on the four–dimensional center manifold. Thus, the existence of neck solutions can be proved.

The last example (Ch. 11) deals with the so-called nonlinear Saint–Venant problem. In fact, most of the present work was developed in order to handle this problem. It is concerned with static elastic deformations of a very long beam loaded only at its ends. The Euler–Lagange equations are the equilibrium equations of nonlinear elasticity, a strongly elliptic system of three elliptic partial differential equations in three space dimensions. It is shown in [Mi88c] that, for this system, a twelve–dimensional center manifold exists which contains solutions corresponding exactly to the solutions of the classical rod equations given in [Ki59, An72, KMS88]. Moreover, the invariance of the problem under rigid–body transformations is now interpreted as the invariance under the action of the (six–dimensional) Lie group of all Euclidian transformations in $\mathbb{R}^3$. The undeformed straight beam is a relative equilibrium, since the cross–section moves linearly with the axial variable.

For hyperelastic materials the equilibrium equations are variational and the functional is the stored–energy function $W_{\text{beam}}(\nabla_{(y,t)}u)$. Applying the Lagrangian reduction procedure we are able to construct a reduced variational problem on the tangent bundle of G, giving rise to a hyperelastic rod model with a reduced energy functional for the rod W_{rod} which is deduced from W_{beam} in a mathematically rigorous way. Moreover, we show that, on the quadratic level, a natural reduction can be achieved, i.e. W_{rod} can be chosen such that the energy of a solution calculated in the rod model gives the same value, up to terms of third order in the strains, as the true energy $\int_\Sigma W_{\text{beam}} dy$ of the associated beam solution.

Part I

Hamiltonian and Lagrangian theory

Chapter 2

Notations and basic facts on center manifolds

We consider a possibly *infinite-dimensional manifold* $\mathcal{X}$ which is modelled over a reflexive Banach space X ([We71, La62]). As our theory is local with respect to some base point we will often identify $\mathcal{X}$ with X by a local coordinate system. As a general reference for infinite-dimensional Hamiltonian systems we use [CM74, AM78, Ma81] and [HM83, Ch.5.3]. Throughout the whole work we try to be as self-contained as possible. Therefore we avoid the extensive use of special notations and facts in the calculus on manifolds. This should enable readers, who are not familiar with this field, to follow the analysis. However, at some points this has to be payed with less consistent notations or with less elegant proofs.

As usual the *tangent* and the *cotangent bundles* $T\mathcal{X}$ and $T^*\mathcal{X}$ are defined as the unions of the local tangent and cotangent spaces $(x, T_x\mathcal{X})$ and $(x, T_x^*\mathcal{X})$, respectively. Hence, $T\mathcal{X}$ and $T^*\mathcal{X}$ are locally isomorphic to $X \times X$ and $X \times X^*$, respectively, where X^* denotes the dual space of X consisting of all continuous linear forms on X with the natural contraction written as $\langle \cdot, \cdot \rangle_X : X^* \times X \to \mathbb{R}$.

All manifolds, functions, mappings, and bundles over manifolds will only be assumed to have a finite order of differentiability, which is sufficiently high to perform the calculations. A theory of C^k–manifolds is, for instance, developed in [La62]. We cannot work in the C^∞–setting as the center manifold, being our main interest, is in general not a C^∞–manifold (see [CH82]). We will not specify the order of differentiability explicitly in each point. From the context it will be clear what regularity is needed in a certain step. For instance, on a C^k–manifold, being defined to have charts ϕ_i such that the compositions $\phi_i \circ \phi_j^{-1}$ are in C^k, we may define only C^k–functions, the tangent space is a C^{k-1}–manifold, and so on.

Throughout the whole work it suffices to consider C^6–manifolds. Starting with a