PART A

Studies in Applied Mechanics

37

Constitutive Equations for Engineering Materials

Volume 1: Elasticity and Modeling

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Elsevier

STUDIES IN APPLIED MECHANICS 37A

Constitutive Equations for Engineering Materials

Volume 1:
Elasticity and Modeling

Revised Edition

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ELSEVIER
Amsterdam – London – New York – Tokyo 1994

ELSEVIER SCIENCE B.V.
Sara Burgerhartstraat 25
P.O. Box 211, 1000 AE Amsterdam, The Netherlands

Constitutive Equations for Engineering Materials
Volume 1: Elasticity and Modeling (W. F. Chen and A. F. Saleeb)
First edition 1982 (John Wiley & Sons, Inc., New York, USA)
Second, revised edition 1994

Library of Congress Cataloging-in-Publication Data

Chen, Wai-Fah, 1936-
Constitutive equations for engineering materials / Wai-Fah Chen, Atef F. Saleeb.
p. cm. -- (Studies in applied mechanics ; v. 37A-37B)
Includes bibliographical references and index.
Contents: v. 1. Elasticity and modeling -- v. 2. Plasticity and modeling.
ISBN 0-444-88408-4
1. Structures, Theory of. 2. Elastic analysis (Engineering) 3. Plastic analysis (Engineering) 4. Continuum mechanics. I. Saleeb, Atef F. (Atef Fatthy), 1952- . II. Title. III. Series.
TA653.C48 1994
624.1--dc20 94-9081
CIP

ISBN: 0-444-88408-4

This book is printed on acid-free paper.

Printed in The Netherlands

STUDIES IN APPLIED MECHANICS 37A

Constitutive Equations for Engineering Materials

Volume 1: Elasticity and Modeling

To
Lily Chen

Ziza Saleeb

PREFACE TO THE SECOND EDITION

The first edition of Volume 1 of this two-volume book was published by Wiley-Interscience, New York in 1982 and was out-of-print five years later. The history of writing Volume 2 of this book extends over the past 12 years and in the meantime the whole text of Volume 1 has been re-examined, and many minor improvements have been made by correcting errors, improving quality of some figures and adding some new problems at the end of the first three chapters. Some of these improvements have resulted from our experience of giving courses at Purdue University and elsewhere over the past 12 years and many valuable suggestions and corrections have come from numerous students and correspondents to whom we are grateful.

The many recent developments in the theory of plasticity and its applications to concrete and soil mechanics problems which have occurred since the first edition of Volume 1 on *Elasticity and Modeling* was written are now reflected in the last four chapters of Volume 2 on *Plasticity and Modeling*. The time has come to publish both volumes together for the first time that will provide engineers and research workers the unified approach to information and background needed to enable them to follow the latest developments in this field. The arrangement of the present edition of Volume 1 remains the same as the first edition and the arrangement of Volume 2 follows closely the presentation and organization of Volume 1.

W.F. CHEN

July, 1993
West Lafayette, IN

PREFACE

The initial draft of this book grew out of lectures that Professor Chen gave for a number of years to graduate students in civil engineering at Lehigh University and Purdue University. Originally, the book was aimed squarely at structural engineers. Its purpose was in part to discuss the theories of elasticity and plasticity in a form that did not require extensive mathematical experience beyond the usual background of a structural engineer, and in part to provide the necessary foundation of such mathematical theories for finite element applications in structural engineering and structural mechanics. However, important results in extending these theories to model the constitutive behavior of nonmetallic materials such as concrete and soil were reported in the literature while the final draft was being prepared. The inclusion of these results and the revision of many sections and chapters in the light which these results throw on the relation between the theories for metals and their generalization for concrete and soil have resulted in a broader scope and new objectives for the book. Now its purpose is in part to discuss modern methods of constitutive modeling of engineering materials based on the principles of elasticity and plasticity in a form that is suitable for a civil engineer, and in part to provide a compact and convenient state-of-the-art summary of such mathematical modeling techniques for material behavior in nonlinear finite element analysis for civil engineers in general, and structural, materials, and geotechnical engineers in particular.

The book is intended as a text as well as a reference book for self-study. The reader should have a basic background in mechanics, strength of materials, calculus, and material behavior of metals, concrete, and soil which are normally covered for students in civil engineering. In Parts Two and Three of Volume 1, and Parts Three and Four of Volume 2, some exposure to mechanics of reinforced concrete, soil mechanics, and finite element methods will be helpful.

Throughout the book, there are many illustrative examples. Some show numerical work leading to results illustrating the physical content of the formulations. Others let the reader come to grips with mathematical techniques that are often used in the subsequent derivations and formulations. Further, the reader can check his mastery of the subject on many problems given at the end of the first few chapters of each volume.

We have benefited greatly from the research project on "Constitutive Modeling and Earthquake-Induced Landslides" sponsored by the National Science Foundation for 1979–1981, when the book began to take shape. The preparation of technical reports for this research project inspired us to transform them into a part of a textbook for the graduate student in civil engineering. Without the inclusion of these results, the account of the theories of elasticity and plasticity would have remained rather incomplete for finite element applications in civil engineering. The program manager of this research project is Dr. W. W. Hakala and the fellow workers are Professor S. L. Koh and Messrs. S. W. Chan, C. J. Chang, M. F. Chang, S. S. Hsieh, and E. Mizuno. Their contributions to this research project are gratefully acknowledged.

Professor Chen wishes to extend his thanks to Professors H. L. Michael and M. B. Scott of Purdue University for their continuing support over many years, and to many of his colleagues and students in the structures area of the School of Civil Engineering for their help and encouragement during his writing.

We thank the secretarial staff of the structures and geotechnical areas of the School of Civil Engineering of Purdue University for the careful preparation of various parts of the manuscript.

W. F. CHEN
A. F. SALEEB

West Lafayette, Indiana
December 1981

NOTATION

Given below is a list of the principal symbols and notations used in the book. All notations and symbols are defined in the text when they first appear. Symbols which have more than one meaning are defined clearly when used to avoid confusion, and usually the correct meaning will be obvious from the context.

Stresses and Strains

$\sigma_1, \sigma_2, \sigma_3$	Principal stresses
σ_{ij}	Stress tensor
s_{ij}	Stress deviator tensor
σ	Normal stress
τ	Shear stress
$\sigma_{\text{oct}} = \frac{1}{3} I_1$	Octahedral normal stress
$\tau_{\text{oct}} = \sqrt{\frac{2}{3} J_2}$	Octahedral shear stress
$\sigma_m = \sigma_{\text{oct}}$	Mean normal (hydrostatic) stress
$\tau_m = \sqrt{\frac{2}{5} J_2}$	Mean shear stress
s_1, s_2, s_3	Principal stress deviators
$\varepsilon_1, \varepsilon_2, \varepsilon_3$	Principal strains
ε_{ij}	Strain tensor
e_{ij}	Strain deviator tensor
ε	Normal strain
γ	Engineering shear strain
$\varepsilon_v = I'_1$	Volumetric strain
$\varepsilon_{\text{oct}} = \frac{1}{3} I'_1$	Octahedral normal strain
$\gamma_{\text{oct}} = 2\sqrt{\frac{2}{3} J'_2}$	Octahedral engineering shear strain
e_1, e_2, e_3	Principal strain deviators

Invariants

$I_1 = \sigma_1 + \sigma_2 + \sigma_3 = \sigma_{ii} =$ first invariant of stress tensor

$J_2 = \frac{1}{2} s_{ij} s_{ij}$

$= \frac{1}{6}[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2] + \tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2$

$=$ second invariant of stress deviator tensor

$J_3 = \frac{1}{3} s_{ij} s_{jk} s_{ki} =$ third invariant of stress deviator tensor

$\cos 3\theta = \dfrac{3\sqrt{3}}{2} \dfrac{J_3}{J_2^{3/2}}$ where θ is the angle of similarity defined in Figure 5.13

$I_1' = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon_v =$ first invariant of strain tensor

$\rho = \sqrt{2J_2} =$ deviatoric length defined in Figure 5.12

$\xi = \dfrac{1}{\sqrt{3}} I_1 =$ hydrostatic length defined in Figure 5.12

$$\begin{aligned} J_2' &= \tfrac{1}{2} e_{ij} e_{ij} \\ &= \tfrac{1}{6}[(\varepsilon_x - \varepsilon_y)^2 + (\varepsilon_y - \varepsilon_z)^2 + (\varepsilon_z - \varepsilon_x)^2] + \varepsilon_{xy}^2 + \varepsilon_{yz}^2 + \varepsilon_{zx}^2 \\ &= \text{second invariant of strain deviator tensor} \end{aligned}$$

Material Parameters

f_c'	Uniaxial compressive cylinder strength ($f_c' > 0$)
f_t'	Uniaxial tensile strength
f_{bc}'	Equal biaxial compressive strength ($f_{bc}' > 0$)
E	Young's modulus
ν	Poisson's ratio
$K =$	$\dfrac{E}{3(1-2\nu)} =$ Bulk modulus
$G =$	$\dfrac{E}{2(1+\nu)} =$ Shear modulus
c, ϕ	Cohesion and friction angle in Mohr–Coulomb criterion
α, k	Constants in Drucker–Prager criterion
k	Yield (failure) stress in pure shear

Miscellaneous

$\{\ \}$	Vector
$[\ ]$	Matrix
C_{ijkl}	Material stiffness tensor
D_{ijkl}	Material compliance tensor
$f(\ )$	Failure criterion or yield function
x, y, z or x_1, x_2, x_3	Cartesian coordinates
δ_{ij}	Kronecker delta
$W(\varepsilon_{ij})$	Strain energy density
$\Omega(\sigma_{ij})$	Complementary energy density
$l_{ij} =$	$\cos(x_i', x_j) =$ The cosines of the angles between x_i' and x_j axes (see Section 1.11)
ε_{ijk}	Alternating tensor defined in Section 1.10

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37A. Constitutive Equations for Engineering Materials
Volume 1: Elasticity and Modeling (Chen and Saleeb)

37B. Constitutive Equations for Engineering Materials
Volume 2: Plasticity and Modeling (Chen)

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CONTENTS

VOLUME ONE

VOLUME TWO

Introduction

In outline form, the solution of a solid mechanics problem at each instant of time must satisfy:

1. Equations of equilibrium or of motion.
2. Conditions of geometry or the compatibility of strains and displacements.
3. Material constitutive laws or stress–strain relations.

From considerations of equilibrium (or of motion), one can relate the stresses inside a body to the body forces and external forces acting on the surface of a body. There are three *equations of equilibrium* relating the six components of stress tensor for an infinitesimal element of the body. In linear problems these equations do not contain strains or displacements; in nonlinear problems they often do. In problems of dynamics, the equilibrium equations are replaced by the equations of motion, which contain second-order time derivatives of the displacements. These are the first set of equations.

From considerations of geometry or kinematics, one can relate the strains inside a body to the displacements of a body. There are six *equations of kinematics* expressing the six components of strain tensor in terms of the three components of displacements; they are known as the *strain-displacement relations*. These are the second set of equations.

Clearly, both the equations of equilibrium and the equations of kinematics are independent of the particular material of which the body is made. The influence of this material is expressed by a third set of equations, the *constitutive equations*. They describe the relations between stresses and strains. In the simplest case, there are six equations expressing the strain components in terms of stress components, or vice versa. If they are linear, they are known as Hooke's law.

The six stress components, six strain components, and three displacement components are connected by the three equilibrium equations, six kinematic equations, and six constitutive equations. These 15 unknown quantities of stresses, strains, and displacements inside a body are determined from the system of 15 equations expressing laws of nature.

For a long time, mechanics of deformable solids has been based upon Hooke's law of linear elasticity for describing material behavior because of its simplicity. It is well known that most civil engineering materials such as metals, concrete, wood, soil, and rock are not linearly elastic for the entire range of loading of practical interest. In fact, actual behavior of these materials is very complicated and they show a great variety of behavior when subjected to different conditions. Drastic idealizations are therefore essential in order to develop simple mathematical models for practical applications.

No one mathematical model can completely describe the complex behavior of real materials under all conditions. Each material model is aimed at a certain class of phenomena, captures their essential features, and disregards what is considered to be of minor importance in that class of applications. Thus this constitutive model meets its limits of applicability where a disregarded influence becomes important. As an example, Hooke's law has been used successfully in structural and geotechnical engineering to describe the general behavior of a structure or soil media under short-term working load conditions, but it fails to predict the behavior and strength of a structure or a soil–structure interaction problem near ultimate strength conditions, because plastic deformation at this load level attains a dominating influence, whereas elastic deformation becomes of minor importance.

Volume 1 — Elasticity and Modeling

For some materials, their behavior may be idealized as *time independent*, where the effects of time can be neglected. This time-independent behavior of materials can be further idealized as *elastic* behavior and *plastic* behavior. For an elastic material there exists a one-to-one coordination between stress and strain. Thus a body that consists of this material returns to its original shape whenever all stresses are reduced to zero. This reversibility is not the case for a plastic material. The elasticity-based constitutive models for civil engineering materials are the subject of Volume 1 of this book. Volume 2 of this book presents the plasticity-based constitutive models.

In a more restricted sense, an elastic material must also satisfy the energy equation of thermodynamics. The elastic material characterized by this additional requirement is known as *hyperelastic*. On the other hand, the minimal requirement for a material to qualify as elastic in any sense is that there exists a one-to-one coordination between stress increment and strain increment. Thus a body that consists of this material returns to its original state of deformation whenever all stress increments are reduced to zero. This reversibility in the infinitesimal sense justifies the use of the term *hypoelastic* for elastic materials satisfying only this minimal requirement. The incremental constitutive formulations based on hypoelastic models have been increasingly used in recent years by structural and geotechnical engineers for materials such as metals, concrete, wood, and soil, in which the state of stress is generally a function of the current state of strain as well as of the stress path followed to reach that state.

Volume 1 of this book presents the constitutive equations of hypoelastic, elastic, and hyperelastic materials which possess the hallmark of elasticity in increasing measure. The volume is divided into three parts containing seven chapters. The first part, Chapters 1 to 4, is concerned with the basic concepts in elasticity. The necessary concepts and notations of vector and tensor analysis are first developed in Chapter 1, since these are not familiar terms for civil engineers. The next two chapters are concerned with the concepts of stress and strain, and develop the basic equations of equilibrium and kinematics which can be discussed without assuming a specific constitutive equation. Basic assumptions of elastic stress–strain relations are plausibly explained in Chapter 4, and mathematical and physical reasonings are used to derive *general* constitutive equations from them, first in tensor forms, and then in matrix forms that are deemed of great interest for engineering applications.

The second part, Chapters 5 and 6, is devoted to the specific formulations and applications of the elasticity-based models to reinforced concrete materials. There are two major sources of nonlinearity in reinforced concrete: cracking of the concrete, and plasticity of the reinforcement and of the compression concrete. The nonlinearity due to cracking and the failure criteria of concrete are considered in Chapter 5, and the nonlinearity due to plasticity of the compression concrete is treated in Chapter 6. Examples of using these constitutive models for finite element analysis of typical reinforced concrete structures are also given in this part.

The third part, Chapter 7, is devoted to soil elasticity and failure criteria. Here, as in Part Two, the specific formulations and applications of the elasticity-based models to soil type of media are developed in details. The procedures for fitting various models to a given set of experiments together with their computer implementation and typical finite element applications in geotechnical engineering are plausibly explained and compared with available experimental results.

Volume 2 — Plasticity and Modeling

Volume 2 of this book is concerned with the constitutive equations of plastic materials. It represents a necessary extension of elastic stress–strain relations into the plastic range at which *permanent plastic* strain is possible in addition to elastic strain. This plastic strain remains when the stresses are removed. Thus the strain in a plastic material may be considered as the sum of the reversible *elastic strain* and the permanent irreversible *plastic strain*. Since an elastic stress–strain law as described in Volume 1 is assumed to provide the relation between the incremental changes of stress and elastic strain, the stress–strain law for a plastic material reduces, essentially, to a relation involving the current states of stress and strain and the incremental changes of stress and plastic strain. This relation is generally assumed to be homogeneous and linear in the incremental changes of the components of stress and plastic strain. This assumption precludes viscosity effects, and thus constitutes the time-independent idealization.

The first step toward such a mathematical model is to establish the *yield* limit of an elastic material. This is known as the *yield function*, which is a certain function of the stress components. A plastic material is called *perfectly plastic* or *work hardening* or *work softening* according to whether the yield function as represented by a certain hypersurface in six-dimensional stress space is a fixed surface or admits changes (expansion or contraction) as plastic strain develops. For moderate strains, mild steel behaves approximately as a perfectly plastic material. It is therefore not surprising that in early years (1950–1965) this perfect plasticity model was used almost exclusively and extensively in the analysis and design of steel structures because of its simplicity. The general theorems of limit analysis, developed on the basis of perfect plasticity, furnish simple, direct, and realistic estimates of the load-carrying capacity of these structures.

With the present state of development of finite element computer programs and the increasing use of these programs in recent years for obtaining solutions of practical problems in structural and geotechnical engineering, there is an urgent need for the development of three-dimensional stress–strain models for metals, concrete, and soil based on the principles of plasticity as well as elasticity. For the case of plasticity, this calls for the development of material models which apply to work hardening and/or work softening materials for metals, concrete, and soil. The constitutive equations for these materials may be time dependent or time independent. Together the two volumes develop the time-independent models of elasticity and plasticity. The theory of plasticity as applied to metals, concrete, and soil together with its computer implementation procedures is the subject of Volume 2. For metals at elevated temperature or concrete under sustained long-term loading, however, they show a pronounced influence of time on the deformations. This is known as *creep*. Such a behavior of materials is called *viscoelastic*. The theory of viscoelasticity is beyond the scope of this book.

General Comments

The two-volume book has a twofold aim: (1) to provide the necessary foundations of the theory of elasticity and plasticity for civil engineers in general, and structural, materials, and geotechnical engineers in particular; and (2) to present recent results in the development of constitutive models for metals, concrete, and soil, their numerical implementation to a computer program, and some finite element solutions for typical problems in structural and geotechnical engineering applications. This book is prepared specifically for the benefit of civil engineers who do not specialize in this field; yet there is a great demand on them to apply these mathematical models to their fast-changing