Lecture Notes in Mathematics

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Geometry and Nonlinear Analysis in Banach Spaces

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Berlin Heidelberg New York Tokyo

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Mathematics Subject Classification (1980): primary: 46B20, 58B10, 58C25
secondary: 41A65

ISBN 3-540-15237-7 Springer-Verlag Berlin Heidelberg New York Tokyo
ISBN 0-387-15237-7 Springer-Verlag New York Heidelberg Berlin Tokyo

Printed in Germany

Printing and binding: Beltz Offsetdruck, Hemsbach/Bergstr.
2146/3140-543210

Lecture Notes in Mathematics

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CONTENTS

INTRODUCTION

This monograph is based in part on a series of lectures given by the first author at Cleveland State University and at Texas A & M University since 1980, and in part on the lectures given by the second author at Dalhousie University and the Australian National University since 1981. It concerns the development of certain topics in differential nonlinear analysis in infinite dimensional real Banach spaces. Our motivation derives from the rich and elegant theory of nonlinear analysis in finite dimensional setting and from the fact that the powerful theorems of Stone-Weierstrass, Whitney and Bernstein do not extend to infinite dimensional Banach spaces in general. There seems to be no comprehensive discussion available in the literature on the topics dealt with in these notes. During the last three decades many mathematicians have contributed to various problems on nonlinear analysis in Banach spaces which lie scattered in various journals. A substantial part of these contributions concern the unravelling of the geometric structure of infinite dimensional Banach spaces and the progress that has been made in applying these results to solve problems in nonlinear analysis on such spaces.

There are several excellent monographs on differential analysis in Banach spaces and the related theory of differentiable manifolds modelled on Banach spaces. In this connection we mention the books by Abraham and Robbin [1], Berger [5], Dieudonné [14], Lang [38] and Yamamuro [67] . While we refer to some of these contributions, we minimize overlap with the material in these works.

The nonlinear analysis in infinite dimensional Banach spaces dealt with in these notes is essentially concerned with functions of class C^k, i.e., k times continously Fréchet differentiable functions, for $k \geqslant 1$. In Chapter 1 we recall some basic definitions of smooth functions on open subsets of Banach spaces, some useful convexity properties of Banach spaces, the concept of finite representability, ultraproducts and a few inequalities concerning differentiable functions.

In Chapter 2 we provide a classification of Banach spaces based on the order of differentiability of the norm. We introduce C^k- , BF^k- and UF^k- smooth Banach spaces and discuss their interrelations. A differential characterization of Hilbert spaces modulo isomorphism and an isomorphic classification of superreflexive spaces are given.

In Chapter 3 we discuss various results of Bonic and Frampton [9], Torunczyk [62] and Wells [64] on smooth partitions of unity. We conclude the chapter with a nonlinear characterisation of superreflexive spaces and present a few applications of the characterization to differential analysis and approximation theory.

Chapter 4 is mainly concerned with the extensions of the well known theorems of Bernstein and Whitney and related theorems in the finite dimensional setting to infinite dimensional Banach spaces. The work of Aron and Prolla [4], Nachbin [44], Kurzweil [37] and Restrapo [53] on approximation by differentiable and analytic functions, is discussed. Some recent results of Moulis [43]and Heble [25] on simultaneous approximations by differentiable functions on certain smooth Banach spaces are stated.

The volume concludes with an appendix on differentiable manifolds modelled on Banach spaces, dealing with the diffeomorphism and embedding theorems of Bessaga [6] and Eells-Elworthy [21], and the generalisation of Palais of Morse's theorem on the behaviour of a smooth function in the neighborhood of a nondegenerate critical point to infinite dimensional Banach spaces.

The first author acknowledges his gratitude to Victor Klee for suggesting certain problems concerning C^k-norms on Banach spaces and to him and R.R.Phelps for valuable discussions. The authors express their thanks to Richard Aron for carefully reading a first version of the chapters and making valuable suggestions. They are grateful to the referee for comments and suggestions for improvement. Further they acknowledge their gratitude for the facilities provided by Cleveland State University and Dalhousie University in carrying out this project. The first author wishes to thank Elton Lacey for providing an opportunity to deliver some lectures on the topics dealt with in these notes in the Banach Space Seminars at Texas A & M University during the Spring semester of 1981. The second author acknowledges support of a grant (A 5615) from NSERC(Canada) and also takes this opportunity to thank the Research School of the Australian National University, Canberra, for facilities afforded in 1984.

Finally the authors express their appreciation and thanks to the Editors of the LECTURE NOTES Series for their advice and encouragement.

Chapter 1

Basic Definitions and Geometric Properties

In this introductory chapter we introduce the basic notations and definitions, and recall certain geometric properties essential to our discussion.

All Banach spaces considered here are over the real field R. If $(E, \|\ \|)$ is a Banach space then the conjugate of E is denoted by E^*, and $\|\ \|^*$ is the norm conjugate of the norm $\|\ \|$. If E, F are two Banach spaces then $L(E,F)$ is the Banach space of continuous linear operators on E into F with the supremum norm. The Banach space of continuous k-linear operators T^k on E into F is denoted by $B^k(E,F)$ with the norm defined by

$$\|T^k\| = \sup\{\|T^k(x_1,x_2,\ldots,x_k)\| \mid x_i \in E, \quad \|x_i\| \leq 1\}$$

where $T^k \in B^k(E,F)$. When F is the one dimensional space R the space $B^k(E,F)$ is simply denoted by $B^k(E)$.

1.1 Some Geometric Properties of Banach Spaces

1.1.1 Convexity and smoothness properties

A Banach space $(E, \|\ \|)$ is said to be strictly convex if $x,y \in E$, $x \neq y$, and $\|x\| = \|y\| = 1$ imply $\|\frac{x+y}{2}\| < 1$. E is uniformly convex, if given any $\epsilon > 0$ there is a $\delta(\epsilon) > 0$, such that $\|x\| = \|y\| = 1$, $\|x-y\| \geq \epsilon$ imply $\|\frac{x+y}{2}\| \leq 1 - \delta(\epsilon)$. If $x \in E$, $\|x\| = 1$, E is said to be smooth at x, if there is a unique $\ell_x \in E^*$ with $\|\ell_x\| = 1$, such that $\ell_x(x) = 1$. $\ell_x \in E^*$ is called the support functional for the unit ball U of E at x and $\ell_x^{-1}(1)$ is the hyperplane of support for U at x. It is well known that U is smooth at x iff

(a) $$\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t} = g_x(y) \text{ exists for all } y \in E.$$

Further the limit in (a) exists iff the limit with x replaced by λx, $\lambda \neq 0$ exists. Further $g_{\lambda x} = (\text{sign } \lambda) g_x$. The norm functional on E is said to be differentiable (Fréchet) if for each $x \neq 0$ there exists a linear functional $\ell_x \in E^*$ such that

(b) $$\lim_{\|h\| \to 0} \frac{\|x+h\| - \|x\| - \ell_x(h)}{\|h\|} = 0.$$

E is said to be uniformly smooth if the limit in (a) exists uniformly for all (x,y), $\|x\| = 1 = \|y\|$. It is known that E is uniformly smooth if the limit in (b) is uniformly attained for all x, $\|x\| = 1$.

In the following remark we summarize some well known results concerning the preceeding properties.

1.1.2 Remark

It is well known that if the limit $\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t} = g_x(y)$ exists at some $x \neq 0$, for all $y \in E$, then $\ell_x \in E^*$, and $\|\ell_x\| = 1$. Further if the dual of a Banach space E is smooth (strictly convex) iff E is strictly convex (smooth). A Banach space $(E, \| \ \|)$ is uniformly convex (uniformly smooth) iff E^* is uniformly smooth (uniformly convex). For these results see, Day [11].

1.2 Finite representation of a Banach space

A normed linear space E is said to be finitely represented in another normed linear space F in symbols $E \ll F$, if for each $\epsilon > 0$, and each finite dimensional subspace E_0 of E there exists a subspace F_0 of F, depending on E_0 and ϵ, such that there is an isomorphism T on E_0 onto F_0 satisfying $\|T\| \; \|T^{-1}\| \leq 1 + \epsilon$.

A useful tool in the theory of finite representation is the concept of ultraproduct of a normed linear space. Let S be an infinite set, and $\mathcal{U}$ be a nontrivial (free) ultrafilter on S. If f is a bounded real valued function on S, then $\lim_{\mathcal{U}} f$ is by definition the number, $\sup[\lambda | \{t \in x,\ f(t) > \lambda\} \in \mathcal{U}]$. If $(E, \| \;\|)$ is a normed linear space, and f is a bounded E-valued function on S, let $|f| = \lim_{\mathcal{U}} \|f(t)\|$. It is verified that $| \;|$ is a seminorm on the vector space V of bounded E-valued functions on S. The quotient space of E modulo the kernel of the seminorm $| \;|$, equipped with the quotient norm is called the ultrapower of E with respect to the pair $(S, \mathcal{U})$ and we denote this here by $E(S, \mathcal{U})$.

In the discussion to follow the equivalence class determined by a bounded function $f : S \to E$ is simply denoted by $\tilde{f}$. Further for clarity, we write sometimes $\{f(s)\} \in \tilde{f}$.

We summarize here several useful facts concerning ultrapowers. For a detailed account concerning ultrapowers and their ramifications in the structure theory of Banach spaces, we refer to Stern [56], and Krivine [34].

1.2.1 Proposition

The ultrapower $E(S, \mathcal{U})$ of a Banach space E is a Banach space.

For subsequent use in our discussions we recall some basic concepts from [34].

If E is a Banach space, and $x_1, \ldots, x_n \in E$, let $\Phi : R^n \to R$ be defined by $\Phi(\lambda_1, \lambda_2, \ldots, \lambda_n) = \|\sum_{i=1}^{n} \lambda_i x_i\|$. Φ is called a n-type associated with $x_1, x_2, \ldots, x_n$. Since Φ is absolutely homeogeneous, Φ is determined by its values on the set $S_\infty^n = \{(\lambda_1, \lambda_2, \ldots, \lambda_n) \in R^n \mid \sup_{1 \leq i \leq n} |\lambda_i| = 1\}$. The set of all functions Φ associated with n-tuples of points $\{x_i\}_{i=1}^n \subset E$ could be equipped with the topology τ of uniform convergence on S_∞^n.

1.2.2 Lemma

Let E be a Banach space, and $E(S,\mathcal{U})$ be an ultrapower of E. Let $\{\tilde{f}_i\}_{i=1}^n \subset E(S,\mathcal{U})$ and $\{f_i(s)\}_{s\in S} \in f_i$, $1 \le i \le n$. Let Φ_s, Φ be the n-types of $\{f_i(s)\}_1^n$, and $\{f_i\}_1^n$ respectively. Then $\Phi_s \to \Phi$ following the ultrafilter $\mathcal{U}$ in the topology τ of n-types.

Proof

Since the ranges of f_i are bounded subsets of E, there is a number $M > 0$ such that

$$\|f_i(s)\| \le M, \quad |||f_i||| \le M, \quad 1 \le i \le n, \quad s \in S.$$

From the definition of ultrapowers it follows that

(1) $\lim \Phi_s(\lambda_1,\lambda_2,\ldots,\lambda_n) = \Phi(\lambda_1,\lambda_2,\ldots,\lambda_n)$ for $(\lambda_1,\lambda_2,\ldots,\lambda_n) \in R^n$.

Further

$$|\Phi_S(\lambda_1,\lambda_2,\ldots,\lambda_n) - \Phi_S(\lambda_1^1,\lambda_2^1,\ldots,\lambda_n^1)|$$
$$\le \|\textstyle\sum(\lambda_i - \lambda_i^1)f_i(s)\|$$
$$\le k\sum_{i=1}^n |\lambda_i - \lambda_i^1|.$$

Thus $\{\Phi_s\}_{s\in S}$ is an equicontinuous family of functions on the compact set S_∞^n. Thus by Arzela's theorem it follows that $\lim_{\mathcal{U}} \Phi_s(\lambda_1,\lambda_2,\ldots,\lambda_n) = \Psi(\lambda_1,\ldots,\lambda_n)$ for some continuous function Ψ on $S_\infty^n \to R$, uniformly over S_∞^n. From (1) $\Psi = \Phi$. Thus $\lim \Phi_s = \Phi$ uniformly over S_∞^n.

The main theorem which establishes the importance of the concept of ultrapowers of Banach spaces in the theory of finite representation is the following. Since the proof illustrates a technique we provide the details here.

1.2.3 Theorem

Let E, F be two Banach spaces. Then $E<<F$ iff E is isometric with a subspace of an ultrapower of F.

Proof

Let $E<<F$. Let $\mathcal{F}$ be the set of all finite dimensional subspaces

of E. Consider the free ultrafilter $\mathcal{U}_1$ generated by the tails $\{L \in \mathcal{F} \mid L \supset L_0\}$ where L_0 is an arbitrary subspace of E in $\mathcal{F}$. Let I be the set of positive integers, and $\mathcal{U}_2$ be a free ultrafilter of I. Let $\mathcal{U}$ be the ultrafilter on $S = \mathcal{F} \times I$ generated by the product $\mathcal{U}_1 \times \mathcal{U}_2$. Let $T: E \to F(S, \mathcal{U})$ be defined by setting $T_x = T_L^n(x)$ if $x \in L$, and $Tx = 0$ otherwise, where T_L^n is an isomorphism on $L \to F$ such that $\|T_L^n\| \cdot \|(T_L^n)^{-1}\| \leq 1 + \frac{1}{n}$. Since $\|Tx\| = \lim_{(L,n)\in\mathcal{U}} \|T_L^n(x)\|$, it follows that $\|Tx\| = \|x\|$. We complete the proof by showing that the ultrapower $F(S, \mathcal{U}) << F$. Let G be a finite dimensional subspace of $F(S, \mathcal{U})$, and $\{\tilde{f}_i\}_{i=1}^n$ be a basis for G. Let $\{f_i(s)\}_{s\in S} \in \tilde{f}_i$, $1 \leq i \leq n$. Since $|||\Sigma\lambda_i\tilde{f}_i|||$, $\sup_{1\leq i\leq n} |\lambda_i|$, are norms on R^n, it follows that there exists a constant k such that

$$k\ |||\Sigma\lambda_i\tilde{f}_i||| \geq \sup_{1\leq i\leq n} |\lambda_i|, \text{ for all points } (\lambda_1, \lambda_2, \ldots, \lambda_n) \in R^n.$$

Now if $\epsilon > 0$ from the preceeding lemma it follows that there is a set $Q \subset S$, $Q \in \mathcal{U}$ such that $\left|\ |||\Sigma\lambda_i f_i||| - \|\Sigma\lambda_i f_i(s)\|\ \right| \leq \epsilon$ for all $s \in Q$, and for all $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in R^n$. Let $T: G \to F$ be defined by $T(\Sigma\lambda_i\tilde{f}_i) = \Sigma\lambda_i f_i(s_0)$ where s_0 is a fixed point in Q. If $T(G) = M$, it follows that T is an isomorphism on G onto M, and $\|T\| \cdot \|T^{-1}\| \leq (1 + k\epsilon)/(1 - k\epsilon)$.

1.2.4 <u>Definition</u>

A Banach space E is superreflexive if every Banach space F finitely represented in E is reflexive, James [28]. For an account of superreflexive spaces see VanDulst [63] and Diestel [12].

1.2.5 <u>Theorem</u> [James and Enflo]

The following statements concerning a Banach space E are equivalent:

(a) E is superreflexive.

(b) E is isomorphic to a uniformly smooth (uniformly convex) Banach space F.

(c) E is isomorphic to a Banach space F which is uniformly smooth and uniformly convex.

For a proof of the theorem we refer to [63].

1.3 Multilinear forms and differential concepts in Banach spaces

1.3.1 Remarks on multilinear forms

If T^n is a continuous n-linear form on a Banach space E into R, we already noted in 1.1 that the norm of T^n is given by

$$\sup_{\substack{\|x_i\| = 1 \\ 1 \le i \le n}} |T^n(x_1, x_2, \ldots, x_n)|.$$

If T^n is also symmetric then by the polarization identity, Alexiewicz and Orlicz [2], it follows that

$$T^n(x_1, x_2, \ldots, x_n) = \frac{1}{n!} \sum_{\epsilon_1, \epsilon_2, \ldots, \epsilon_n = 0}^{1} (-1)^{n - \sum_{i=1}^{n} \epsilon_i} T^n\left(\sum_{i=1}^{n} \epsilon_i x_i\right)^{(n)},$$

where $T^n(h)^{(n)} = T^n(h, h, \ldots, h)$. Hence it follows that

$$\|T^n\| \le \frac{2^n n^n}{n!} \sup_{\|x\|=1} |T^n(x, x, \ldots, x)| \le \frac{2^n n^2}{n!} \|T^n\| \quad \ldots \quad (A)$$

The definition of a polynomial in a real variable may be extended to Banach spaces. If T^n is a continuous symmetric n-linear form on a Banach space E, then $T^n(x, x, \ldots, x)$, usually denoted by $T^n(x^{(n)})$, is called a homogeneous polynomial of degree n. A polynomial of degree n on a Banach space E is a function $P(x) = \sum_{i=0}^{n} T^i(x^{(i)})$, where $T^i(x^{(i)})$ is a homogeneous polynomial of degree i, $1 \le i \le n$.

1.3.2 Definition

If U is an open subset of a Banach space E and f is a function on U into a Banach space F then f is said to be differentiable (Fréchet) at $x \in U$ if there is a continuous linear operator T_x on $E \to F$ such that

$$\lim_{\|h\| \to 0} \frac{\|f(x+h) - f(x) - T_x(h)\|}{\|h\|} = 0.$$

T_x is called the derivative of f at x. If f is differentiable at all points $x \in U$, we say that f is differentiable in U, and the map $x \to T_x$ on U into $L(E,F)$ is called the differential of f, and is denoted by Df. If Df is a continuous map then f is called a C^1-function. f is said to be k-times differentiable in U, if $D^{k-1}f$ exists and the map $D^{k-1}f:U \to B^{k-1}(E,F)$ is once differentiable, and in this case the k^{th} differential $D^k f$ is a function on U into $B^k(E,F)$, the Banach space of continuous k-linear operators on E into F. If $D^k f$ is continuous we say f is of class C^k. Further we note that $D^k f(x)$ is a symmetric k-linear operator on E into F. The usual form of Taylor's theorem in the finite dimensional calculus extends to infinite dimensions equally well. Thus if f is a C^k map on U into F, and $x \in U$, then if $[x, x+h] \subset U$,

(A) $$f(x+h) = f(x) + T_x^1(h) + T_x^2(h^{(2)}) + \ldots + T_x^k(h^{(k)}) + \theta_x(h)$$

where (1) T_x^i are continuous symmetric i-linear maps on E into F, (2) the maps $x \to T_x^i$ are continuous on U into $B^i(E,F)$, and

(3) $$\lim_{\|h\| \to 0} \frac{\|\theta_x(h)\|}{\|h\|^k} = 0.$$

In (A) above, $T_x^i(h^{(i)}) = \frac{1}{i!} f^i(x) \cdot h^{(i)}$, $1 \le i \le k$ where $f^i(x)$ is the i^{th} derivative at x. Further sometimes it is useful to use the following form (Lagrange formula) of (A).

$$f(x+h) = f(x) + \sum_{i=1}^{k-1} \frac{1}{i!} f^i(x)(h^{(i)}) + \left(\int_0^1 \frac{(1-s)^{k-1}}{(k-1)!} f^k(x+sh)\,ds\right)h^{(k)}.$$

Further we have the following converse of Taylor's theorem. If $f:U \to F$ is a continuous mapping such that for each $x \in U$,

$$f(x+h) = f(x) + \sum_{i=1}^{k} T_x^i(h^{(i)}) + \theta_x(h)$$

where the maps $x \to T_x^i$ are continuous on U into the space $B^i(E,F)$, with T_x^i symmetric for all $x \in U$, $1 \le i \le k$, and if $\frac{\|\theta_x(h)\|}{\|h\|^k} \to 0$ as $\|h\| \to 0$, then f is a C^k-function on $U \to F$. See [45].

In differential analysis in infinite dimensional Banach spaces it is extremely useful to introduce weaker as well as stronger differentiability of a function than the customary C^k-differentiability as already done here above. We introduce few such concepts here, as they are useful in the discussion to follow.

1.3.3 Definition

Let U be an open subset of a Banach space E and f be a continuous mapping on U into a Banach space F. If $x \in U$, then f is said to be k-times directionally differentiable at x if there are continuous symmetric i-linear transformations T_x^i, $1 \leq i \leq k$ on E into F such that for each $h \in E$

(1) $f(x+th) = f(x) + \sum_{i=1}^{k} t^i T_x^i(h^{(i)}) + \theta_x(th)$ where $\dfrac{\|\theta_x(th)\|}{|t|^k} \to 0$ as $t \to 0$.

It is verified that the i-linear transformations $\{T_x^i\}$, $1 \leq i \leq k$, are uniquely determined. f is said to be k-times Fréchet differentiable (F-differentiable) if the limit in (1) is uniform over the unit sphere, $S_E = \{h | h \in E, \ \|h\| = 1\}$ of E i.e. given $\epsilon > 0$ there is a $\delta > 0$ independent of $h \in S_E$ such that $\|\theta_x(th)\| \leq \epsilon |t|^k$, if $|t| < \delta$. f is said to be k-times directionally (k-times Fréchet) differentiable over the set U if f is k-times directionally (k-times Fréchet) differentiable over U. f is said to be uniformly k-times Fréchet differentiable over a set $P \subset U$ if for each $x \in P$

(2) $f(x+h) = f(x) + \sum_{i=1}^{k} T_x^i(h^{(i)}) + \theta_x(h)$ where $\{T_x^i\}$, $1 \leq i \leq k$

are as described above, and $\lim_{\|h\| \to 0} \dfrac{\|\theta_x(h)\|}{\|h\|^k} = 0$ uniformly for $x \in P$. If f is k-times Fréchet differentiable at all $x \in P$, then f is said to be boundedly k-times Fréchet differentiable over P if $\sup_{x \in P} \|T_x^k\| < \infty$.

The following proposition follows from the definitions, and Taylor's theorem and its converse.

1.3.4 Proposition

If U is an open subset of a Banach space E, and f is k-times

Fréchet differentiable function on U into a Banach space F then f is k-times directionally differentiable. Further f is a C^k-mapping iff the mappings $x \to T_x^i$ are continuous.

Before concluding the introductory chapter we recall two useful results from differential analysis of functions of a real variable.

1.3.5 Markov's inequality

Let P be a polynomial of degree n in a real variable. If $a < b$ are two real numbers, then

$$\sup_{a \leq t \leq b} |P'(t)| \leq \frac{n^2}{(b-a)} \sup_{a \leq t \leq b} |P(t)| .$$

1.3.6 Inequalities for the derivatives of a function

If f is k-times continuously differentiable real valued function on an open interval $]a,b[\subset R$ and if $\sup_{t \in]a,b[} |f(t)| = M_0$, $\sup_{t \in]a,b[} |f^k(t)| = M_k$, then

$$\sup_{t \in]a,b[} |f^{k-1}(t)| \leq \frac{8^{k-1}}{(b-a)^{k-1}} M_0 + \frac{(b+a)}{2} M_k .$$

For the inequalities in 1.3.5 and 1.3.6, see Todd [60], and Dieudonné [14].

Chapter 2

Smoothness Classification of Banach Spaces

In this chapter we discuss various differentiability properties of the norm in a Banach space. We are primarily interested in higher order differentiability of the norm.

2.1 Differentiability properties of norms

2.1.1 Definition

A Banach space $(E, \| \ \|)$ is said to be D^k-smooth (F^k-smooth) at a point $x \in E$, $x \neq 0$, if the $\| \ \|$ is k-times directionally differentiable at x (k-times Fréchet differentiable at x). E is D^k-smooth (F^k-smooth) if it is D^ksmooth at x (F^k-smooth at x) for all $x \neq 0$.

2.1.2 Remark

It follows from the definitions that if E is F^k-smooth then it is D^k-smooth.

2.1.3 Proposition

Let E be D^k-smooth at x, and (*) $\|x+ty\| = \|x\| + \sum_{i=1}^{k} t^i T_x^i(y^{(i)}) + \theta_x(ty)$, $y \in E$, be the expansion assured by the D^k-smoothness at x. Then

(1) E is D^k-smooth at λx, $\lambda \neq 0$, and $T_{\lambda x}^{\ i} = \frac{(\text{sign } \lambda)^i}{|\lambda|^{i-1}} T_x^i$,

(2) (i) $T_x^2(y,y) \geq 0$ for all $y \in E$, and

(ii) if T_x^2 is considered as a map on $E \to E^*$, range $T_x^2 \subset x^{\perp}$.

Proof

(1) follows at once by noting that $\|\lambda x+ty\| = |\lambda| \ \|x+ty\| =$
$= |\lambda|\{\|x\| + \sum_{i=1}^{k} (\frac{t}{\lambda})^i T_x^i(y^{(i)}) + \theta_x(\frac{ty}{\lambda})\}$, and for a fixed λ, $\lambda \neq 0$,

$$\left|\frac{\theta\lambda_x(ty)}{t^k}\right| \to 0 \quad \text{as} \quad t \to 0.$$

To prove (2) enough to verify that $T_x^2(y,y) \geq 0$, if $\|x\| = 1$, for all $y \in E$, by the preceding part of the proposition. Let $z \in E$,