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A Series of Comprehensive Studies in Mathematics

Daniel Revuz Marc Yor

Continuous Martingales and Brownian Motion

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With 8 Figures

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Preface

This book focuses on the probabilistic theory of Brownian motion. This is a good topic to center a discussion around because Brownian motion is in the intersection of many fundamental classes of processes. It is a continuous martingale, a Gaussian process, a Markov process or more specifically a process with independent increments; it can actually be defined, up to simple transformations, as *the* real-valued, centered process with independent increments and continuous paths. It is therefore no surprise that a vast array of techniques may be successfully applied to its study and we, consequently, chose to organize the book in the following way.

After a first chapter where Brownian motion is introduced, each of the following ones is devoted to a new technique or notion and to some of its applications to Brownian motion. Among these techniques, two are of paramount importance: stochastic calculus, the use of which pervades the whole book and the powerful excursion theory, both of which are introduced in a self-contained fashion and with a minimum of apparatus. They have made much easier the proofs of many results found in the epoch-making book of Itô and McKean: Diffusion Processes and their Sample Paths, Springer (1965).

Furthermore, rather than working towards abstract generality, we have tried to study precisely some important examples and to carry through the computations of the laws of various functionals or random variables. Thus we hope to facilitate the task of the beginner in an area of probability theory which is rapidly evolving. The later chapters of the book however, will hopefully be of interest to the advanced reader.

We strove to offer, at the end of each section, a large selection of exercises, the more challenging being marked with the sign * or even **. On one hand, they should enable the reader to improve his understanding of the notions introduced in the text. On the other hand, they deal with many results without which the text might seem a bit "dry" or incomplete; their inclusion in the text however would have increased forbiddingly the size of the book and deprived the reader of the pleasure of working things out by himself. As it is, the text is written with the assumption that the reader will try a good proportion of them, especially those marked with the sign #, and in a few proofs we even indulged in using the results of foregoing exercises.

The text is practically self-contained but for a few results of measure theory. Besides classical calculus, we only ask the reader to have a good knowledge of basic notions of integration and probability theory such as almost-sure and in the mean convergences, conditional expectations, independence and the like. Chapter 0 contains a few complements on these topics. Moreover the early chapters include some classical material on which the beginner can hone his skills.

Each chapter ends up with notes and comments where, in particular, references and credits are given. In view of the enormous literature which has been devoted to Brownian motion and related topics, we have in no way tried to draw a historical picture of the subject and apologize in advance to those who may feel slighted. Likewise our bibliography is not even remotely complete and leaves out the many papers which deal with the relationships of Brownian motion with other fields of Mathematics such as Potential Theory, Harmonic Analysis, Partial Differential Equations and Geometry. A number of excellent books have been written on these subjects some of which we discuss in the notes and comments.

Finally, it is a pleasure to thank those who have offered useful comments on the first drafts in particular J. Jacod, P.A. Meyer, B. Maisonneuve and J. Pitman. Our special thanks go to J.F. Le Gall who put us straight on an inordinate number of points and Shi Zhan who has helped us with the exercises. Each chapter of this book has been taught a number of times by the authors in the last decade, either in a "Cours de 3° Cycle" in Paris or in "crash courses" on Brownian motion; we would like to seize this opportunity of thanking our audiences for their warm response. Last but not least, Josette Saman a pris une part essentielle dans la préparation matérielle du manuscrit et nous l'en remercions bien vivement.

Paris, October 1990

Daniel Revuz
Marc Yor

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Table of Contents

Chapter 0. Preliminaries

In this chapter, we review a few basic facts, mainly from integration and classical probability theories, which will be used throughout the book without further ado. Some other prerequisites, usually from calculus, which will be used in some special parts are collected in the Appendix at the end of the book.

§ 1. Basic Notation

Throughout the sequel, $\mathbb{N}$ will denote the set of integers, namely, $\mathbb{N} = \{0, 1, \dots\}$, $\mathbb{R}$ the set of real numbers, $\mathbb{Q}$ the set of rational numbers, $\mathbb{C}$ the set of complex numbers. Moreover $\mathbb{R}_+ = [0, \infty[$ and $\mathbb{Q}_+ = \mathbb{Q} \cap \mathbb{R}_+$. By positive we will always mean ≥ 0 and say strictly positive for > 0.

Likewise a real-valued function f defined on an interval of $\mathbb{R}$ is increasing (resp. strictly increasing) if $x < y$ entails $f(x) \leq f(y)$ (resp. $f(x) < f(y)$).

If a, b are real numbers, we write:

$$a \wedge b = \min(a, b), \qquad a \vee b = \max(a, b).$$

If E is a set and f a real valued function on E, we use the notation

$$f^+ = f \vee 0, \qquad f^- = -(f \wedge 0), \qquad |f| = f^+ + f^-,$$
$$\|f\| = \sup_{x \in E} |f(x)|.$$

We will write $a_n \downarrow a (a_n \uparrow a)$ if the sequence (a_n) of real numbers increases (decreases) to a.

If $(E, \mathscr{E})$ and $(F, \mathscr{F})$ are measurable spaces, we write $f \in \mathscr{E}/\mathscr{F}$ to say that the function $f\colon E \to F$ is measurable with respect to $\mathscr{E}$ and $\mathscr{F}$. If $(F, \mathscr{F})$ is the real line endowed with the σ-field of Borel sets, we write simply $f \in \mathscr{E}$ and if, in addition, f is positive, we write $f \in \mathscr{E}_+$. The characteristic function of a set A is written 1_A; thus, the statements $A \in \mathscr{E}$ and $1_A \in \mathscr{E}$ have the same meaning.

If Ω is a set and (f_i), $i \in I$, is a collection of maps from Ω to measurable spaces $(E_i, \mathscr{E}_i)$, the smallest σ-field on Ω for which the f_i's are measurable is denoted by $\sigma(f_i,\ i \in I)$. If $\mathscr{C}$ is a collection of subsets of Ω, then $\sigma(\mathscr{C})$ is the smallest σ-field containing $\mathscr{C}$; we say that $\sigma(\mathscr{C})$ is generated by $\mathscr{C}$. The σ-field $\sigma(f_i,\ i \in I)$ is generated by the family $\mathscr{C} = \{f_i^{-1}(A_i), A_i \in \mathscr{E}_i, i \in I\}$. Finally if $\mathscr{E}_i$, $i \in I$, is a family of σ-fields on Ω, we denote by $\bigvee_i \mathscr{E}_i$ the σ-field generated by $\bigcup_i \mathscr{E}_i$. It is the union of the σ-fields generated by the countable sub-families of $\mathscr{E}_i$, $i \in I$.

A measurable space $(E, \mathscr{E})$ is *separable* if $\mathscr{E}$ is generated by a countable collection of sets. In particular, if E is a LCCB space i.e. a locally compact space with countable basis, the σ-field of its Borel sets is separable; it will often be denoted by $\mathscr{B}(E)$. For instance, $\mathscr{B}(\mathbb{R}^d)$ is the σ-field of Borel subsets of the d-dimensional euclidean space.

For a measure m on $(E, \mathscr{E})$ and $f \in \mathscr{E}$, the integral of f with respect to m, if it makes sense, will be denoted by any of the symbols

$$\int f\,dm, \quad \int f(x)\,dm(x), \quad \int f(x)m(dx), \quad m(f), \quad \langle m, f\rangle,$$

and in case E is a subset of a euclidean space and m is the Lebesgue measure, $\int f(x)\,dx$.

If $(\Omega, \mathscr{F}, P)$ is a probability space, we will as usual use the words random variable and expectation in lieu of measurable function and integral and write

$$E[X] = \int_\Omega X\,dP.$$

We will often write r.v. as shorthand for random variable. The law of the r.v. X, namely the image of P by X will be denoted by P_X or $X(P)$. Two r.v.'s defined on the same space are P-equivalent if they are equal P-a.s.

If $\mathscr{G}$ is a sub-σ-field of $\mathscr{F}$, the conditional expectation of X with respect to $\mathscr{G}$, if it exists, is written $E[X|\mathscr{G}]$. If $X = 1_A$, $A \in \mathscr{F}$, we may write $P(A|\mathscr{G})$. If $\mathscr{G} = \sigma(X_i, i \in I)$ we also write $E[X|X_i, i \in I]$ or $P(A|X_i, i \in I)$. As is well-known conditional expectations are defined up to P-equivalence, but we will often omit the qualifying P-a.s. When we apply conditional expectation successively, we shall abbreviate $E[E[X|\mathscr{F}_1]|\mathscr{F}_2]$ to $E[X|\mathscr{F}_1|\mathscr{F}_2]$.

We recall that if Ω is a Polish space (i.e. a metrizable complete topological space with a countable dense subset), $\mathscr{F}$ the σ-field of its Borel subsets and if $\mathscr{G}$ is separable, then there is a regular conditional probability distribution given $\mathscr{G}$.

If μ and ν are two σ-finite measures on $(E, \mathscr{E})$, we write $\mu \perp \nu$ to mean that they are mutually singular, $\mu \ll \nu$ to mean that μ is absolutely continuous with respect to ν and $\mu \sim \nu$ if they are equivalent, namely if $\mu \ll \nu$ and $\nu \ll \mu$. The Radon-Nikodym derivative of μ with respect to ν is written $\left.\dfrac{d\mu}{d\nu}\right|_{\mathscr{F}}$ and the $\mathscr{F}$ is dropped when there is no risk of confusion.

§2. Monotone Class Theorem

We will use several variants of this theorem which we state here without proof.

(2.1) Theorem. *Let $\mathscr{S}$ be a collection of subsets of Ω such that*
i) $\Omega \in \mathscr{S}$
ii) *if* $A, B \in \mathscr{S}$ *and* $A \subset B$, *then* $B \setminus A \in \mathscr{S}$;
iii) *if* $\{A_n\}$ *is an increasing sequence of elements of* $\mathscr{S}$ *then* $\bigcup A_n \in \mathscr{S}$.
If $\mathscr{S} \supset \mathscr{F}$ *where* $\mathscr{F}$ *is closed under finite intersections then* $\mathscr{S} \supset \sigma(\mathscr{F})$.

The above version deals with sets. We turn to the functional version.

(2.2) Theorem. *Let $\mathscr{H}$ be a vector space of bounded real-valued functions on Ω such that*
i) *the constant functions are in* $\mathscr{H}$,
ii) $\mathscr{H}$ *is closed under uniform convergence,*
iii) *if* $\{h_n\}$ *is an increasing sequence of positive elements of* $\mathscr{H}$ *such that* $h = \sup_n h_n$ *is bounded, then* $h \in \mathscr{H}$.

If $\mathscr{C}$ *is a subset of* $\mathscr{H}$ *which is stable under pointwise multiplication, then* $\mathscr{H}$ *contains all the bounded* $\sigma(\mathscr{C})$*-measurable functions.*

The hypothesis of this theorem may be altered in several ways. For instance the conclusion is still valid if $\mathscr{H}$ is a set of bounded functions satisfying ii) and iii) and $\mathscr{C}$ is an algebra containing the constants; it is also valid if $\mathscr{H}$ is a set of bounded functions satisfying iii) and $\mathscr{C}$ is a vector space, stable under the operations $\wedge$ and $\vee$ and containing the constants. This last version avoids uniform convergence.

The above theorems will be used, especially in Chap. III, in the following set-up. We have a family f_i, $i \in I$, of mappings of a set Ω into measurable spaces $(E_i, \mathscr{E}_i)$. We assume that for each $i \in I$ there is a subclass $\mathscr{N}_i$ of $\mathscr{E}_i$, closed under finite intersections and such that $\sigma(\mathscr{N}_i) = \mathscr{E}_i$. We then have the following results.

(2.3) Theorem. *Let $\mathscr{N}$ be the family of sets of the form $\bigcap_{i \in J} f_i^{-1}(A_i)$ where A_i ranges through $\mathscr{N}_i$ and J ranges through the finite subsets of I; then $\sigma(\mathscr{N}) = \sigma(f_i, i \in I)$.*

(2.4) Theorem. *Let $\mathscr{H}$ be a vector space of real-valued functions on Ω, containing 1_Ω, satisfying property* iii) *of Theorem* (2.2) *and containing all the functions 1_Γ for $\Gamma \in \mathscr{F}$. Then, $\mathscr{H}$ contains all the bounded, real-valued, $\sigma(f_i, i \in I)$-measurable functions.*

§ 3. Completion

If $(E, \mathscr{E})$ is a measurable space and μ a probability measure on $\mathscr{E}$, the completion $\mathscr{E}^\mu$ of $\mathscr{E}$ with respect to μ is the σ-field of subsets B of E such that there exist B_1 and B_2 in $\mathscr{E}$ with $B_1 \subset B \subset B_2$ and $\mu(B_2 \setminus B_1) = 0$. If γ is a family of probability measures on $\mathscr{E}$, the σ-field

$$\mathscr{E}^\gamma = \bigcap_{\mu \in \gamma} \mathscr{E}^\mu$$

is called the completion of $\mathscr{E}$ with respect to γ. If γ is the family of all probability measures on $\mathscr{E}$, then $\mathscr{E}^\gamma$ is denoted by $\mathscr{E}^*$ and is called the σ-field of *universally measurable sets.*

If $\mathscr{F}$ is a sub-σ-algebra of $\mathscr{E}^\gamma$ we define the *completion of $\mathscr{F}$ in $\mathscr{E}^\gamma$ with respect to γ* as the family of sets A with the following property: for each $\mu \in \gamma$, there is a set B such that $A \Delta B$ is in $\mathscr{E}^\gamma$ and $\mu(A \Delta B) = 0$. This family will be denoted $\bar{\mathscr{F}}^\gamma$; the reader will show that it is a σ-field which is larger than $\mathscr{F}^\gamma$. Moreover, it has the following characterization.

(3.1) Proposition. *A set A is in $\bar{\mathscr{F}}^\gamma$ if and only if for every $\mu \in \gamma$ there is a set B_μ in $\mathscr{F}$ and two μ-negligible sets N_μ and M_μ in $\mathscr{E}$ such that*

$$B_\mu \backslash N_\mu \subset A \subset B_\mu \cup M_\mu.$$

Proof. Left to the reader as an exercise. □

The following result gives a means of checking the measurability of functions with respect to σ-algebras of the $\bar{\mathscr{F}}^\gamma$-type.

(3.2) Proposition. *For $i = 1, 2$, let $(E_i, \mathscr{E}_i)$ be a measurable space, γ_i a family of probability measures on $\mathscr{E}_i$ and $\mathscr{F}_i$ a sub-σ-algebra of $\mathscr{E}_i^{\gamma_i}$. If f is a map which is both in $\mathscr{E}_1/\mathscr{E}_2$ and $\mathscr{F}_1/\mathscr{F}_2$ and if $f(\mu) \in \gamma_2$ for every $\mu \in \gamma_1$ then f is in $\bar{\mathscr{F}}_1^{\gamma_1}/\bar{\mathscr{F}}_2^{\gamma_2}$.*

Proof. Let A be in $\bar{\mathscr{F}}_2^{\gamma_2}$. For $\mu \in \gamma_1$, since $\nu = f(\mu)$ is in γ_2, there is a set $B_\nu \in \mathscr{F}_2$ and two ν-negligible sets N_ν and M_ν in $\mathscr{E}_2$ such that

$$B_\nu \backslash N_\nu \subset A \subset B_\nu \cup M_\nu.$$

The set $B_\mu = f^{-1}(B_\nu)$ belongs to $\mathscr{F}_1$, the sets $N_\mu = f^{-1}(N_\nu)$ and $M_\mu = f^{-1}(M_\nu)$ are μ-negligible sets of $\mathscr{E}_1$ and

$$B_\mu \backslash N_\mu \subset f^{-1}(A) \subset B_\mu \cup M_\mu.$$

This entails that $f^{-1}(A) \in \bar{\mathscr{F}}_1^{\gamma_1}$, which completes the proof.

§4. Functions of Finite Variation and Stieltjes Integrals

This section is devoted to a set of properties which will be used constantly throughout the book.

We deal with real-valued, right-continuous functions A with domain $[0, \infty[$. The results may be easily extended to the case of $\mathbb{R}$. The value of A in t is denoted A_t or $A(t)$. Let Δ be a subdivision of the interval $[0, t]$ with $0 = t_0 < t_1 < \cdots < t_n = t$; the number $|\Delta| = \sup_i |t_{i+1} - t_i|$ is called the *modulus* or *mesh* of Δ. We consider the sum

$$S_t^\Delta = \sum_i |A_{t_{i+1}} - A_{t_i}|.$$

If Δ' is another subdivision which is a refinement of Δ, that is, every point t_i of Δ is a point of Δ', then plainly $S_t^{\Delta'} \geq S_t^{\Delta}$.

(4.1) Definition. The function A is of *finite variation* if for every t

$$S_t = \sup_{\Delta} S_t^{\Delta} < +\infty.$$

The function $t \to S_t$ is called the *total variation* of A and S_t is the variation of A on $[0, t]$. The function S is obviously positive and increasing and if $\lim_{t\to\infty} S_t < +\infty$, the function A is said to be of *bounded variation.*

The same notions could be defined on any interval $[a, b]$. We shall say that a function A on the whole line is of finite variation if it is of *finite variation* on any compact interval but not necessarily of bounded variation on the whole of $\mathbb{R}$.

Let us observe that C^1-functions are of finite variation. Monotone finite functions are of finite variation and conversely we have the

(4.2) Proposition. *Any function of finite variation is the difference of two increasing functions.*

Proof. The functions $(S + A)/2$ and $(S - A)/2$ are positive and increasing as the reader can easily show, and A is equal to their difference. □

This decomposition is moreover *minimal* in the sense that if $A = F - G$ where F and G are positive and increasing, then $(S + A)/2 \leq F$ and $(S - A)/2 \leq G$.

As a result, the function A has left limits in any $t \in]0, \infty[$. We write A_{t-} or $A(t-)$ for $\lim_{s\uparrow t} A_s$ and we set $A_{0-} = 0$. We moreover set $\Delta A_t = A_t - A_{t-}$; this is the *jump* of A in t.

The importance of these functions lies in the following

(4.3) Theorem. *There is a one-to-one correspondence between Radon measures μ on $[0, \infty[$ and right-continuous functions A of finite variation given by*

$$A_t = \mu([0, t]).$$

Consequently $A_{t-} = \mu([0, t[)$ and $\Delta A_t = \mu(\{t\})$. Moreover the variation S of A corresponds to the total variation $|\mu|$ of μ and the minimal decomposition of μ into positive and negative parts.

If f is a locally bounded Borel function on $\mathbb{R}_+$, its *Stieltjes integral with respect to A*, denoted

$$\int_0^t f_s\, dA_s, \quad \int_0^t f(s)\, dA(s) \quad \text{or} \quad \int_{]0,t]} f(s)\, dA_s$$

is the integral of f with respect to μ on the interval $]0, t]$. The reader will observe that the jump of A at zero does not come into play and that $\int_0^t dA_s = A_t - A_0$. If we want to consider the integral on $[0, t]$, we will write $\int_{[0,t]} f(s)\, dA_s$. The integral on $]0, t]$ is also denoted by $(f \cdot A)_t$. We point out that the map $t \to (f \cdot A)_t$ is itself a right-continuous function of finite variation.

A consequence of the Radon-Nikodym theorem applied to μ and to the Lebesgue measure λ is the

(4.4) Theorem. *A function A of finite variation is λ-a.e. differentiable and there exists a function B of finite variation such that $B' = 0$ λ-a.e. and*

$$A_t = B_t + \int_0^t A'_s \, ds.$$

The function A is said to be *absolutely continuous* if $B = 0$. The corresponding measure μ is then absolutely continuous with respect to λ.

We now turn to a series of notions and properties which are very useful in handling Stieltjes integrals.

(4.5) Proposition (Integration by parts formula). *If A and B are two functions of finite variation, then for any t,*

$$A_t B_t = A_0 B_0 + \int_0^t A_s \, dB_s + \int_0^t B_{s-} \, dA_s.$$

Proof. If μ (resp. ν) is associated with A (resp. B) both sides of the equality are equal to $(\mu \otimes \nu)([0,t]^2)$; indeed $\int_0^t A_s \, dB_s$ is the measure of the upper triangle including the diagonal, $\int_0^t B_{s-} \, dA_s$ the measure of the lower triangle excluding the diagonal and $A_0 B_0 = \mu \otimes \nu(\{0,0\})$. □

To reestablish the symmetry, the above formula can also be written

$$A_t B_t = \int_0^t A_{s-} \, dB_s + \int_0^t B_{s-} \, dA_s + \sum_{s \leq t} \Delta A_s \Delta B_s.$$

The sum on the right is meaningful as A and B have only countably many discontinuities. In fact, A can be written uniquely $A_t = A_t^c + \sum_{s \leq t} \Delta A_s$ where A^c is continuous and of finite variation.

The next result is a "chain rule" formula.

(4.6) Proposition. *If F is a C^1-function and A is of finite variation, then $F(A)$ is of finite variation and*

$$F(A_t) = F(A_0) + \int_0^t F'(A_{s-}) \, dA_s + \sum_{s \leq t} (F(A_s) - F(A_{s-}) - F'(A_{s-}) \Delta A_s).$$

Proof. The result is true for $F(x) = x$, and if it is true for F it is true for $xF(x)$ as one can deduce from the integration by parts formula; consequently the result is true for polynomials. The proof is completed by approximating a C^1-function by a sequence of polynomials. □

As an application of the notions introduced thus far, let us prove the useful