Lecture Notes in Mathematics 1526

J. Azéma P. A. Meyer M. Yor (Eds.)

Séminaire de Probabilités XXVI

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Editorial Policy

for the publication of proceedings of conferences
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Lecture Notes aim to report new developments - quickly, informally and at a high level. The following describes criteria and procedures for multi-author volumes. For convenience we refer throughout to "proceedings" irrespective of whether the papers were presented at a meeting.

The editors of a volume are strongly advised to inform contributors about these points at an early stage.

§ 1. One (or more) expert participant(s) should act as the scientific editor(s) of the volume. They select the papers which are suitable (cf. §§ 2 - 5) for inclusion in the proceedings, and have them individually refereed (as for a journal). It should not be assumed that the published proceedings must reflect conference events in their entirety. The series editors will normally not interfere with the editing of a particular proceedings volume - except in fairly obvious cases, or on technical matters, such as described in §§ 2 - 5. The names of the scientific editors appear on the cover and title-page of the volume .

§ 2. The proceedings should be reasonably homogeneous i.e. concerned with a limited and welldefined area. Papers that are essentially unrelated to this central topic should be excluded. One or two longer survey articles on recent developments in the field are often very useful additions. A detailed introduction on the subject of the congress is desirable.

§ 3. The final set of manuscripts should have at least 100 pages and preferably not exceed a total of 400 pages . Keeping the size below this bound should be achieved by stricter selection of articles and NOT by imposing an upper limit on the length of the individual papers .

§ 4. The contributions should be of a high mathematical standard and of current interest. Research articles should present new material and not duplicate other papers already published or due to be published. They should contain sufficient background and motivation and they should present proofs, or at least outlines of such, in sufficient detail to enable an expert to complete them. Thus summaries and mere announcements of papers appearing elsewhere cannot be included, although more detailed versions of, for instance, a highly technical contribution may well be published elsewhere later.

Contributions in numerical mathematics may be acceptable without formal theorems/proofs provided they present new algorithms solving problems (previously unsolved or less well solved) or develop innovative qualitative methods, not yet amenable to a more formal treatment.

Surveys, if included, should cover a sufficiently broad topic, and should normally not just review the author's own recent research. In the case of surveys, exceptionally, proofs of results may not be necessary.

§ 5. "Mathematical Reviews" and "Zentralblatt für Mathematik" recommend that papers in proceedings volumes carry an explicit statement that they are in final form and that no similar paper has been or is being submitted elsewhere, if these papers are to be considered for a review. Normally, papers that satisfy the criteria of the Lecture Notes in Mathematics series also satisfy this requirement, but we strongly recommend that each such paper carries the statement explicitly.

§ 6. Proceedings should appear soon after the related meeting. The publisher should therefore receive the complete manuscript (preferably in duplicate) including the Introduction and Table of Contents within nine months of the date of the meeting at the latest.

§ 7. Proposals for proceedings volumes should be sent to one of the editors of the series or to Springer-Verlag Heidelberg. They should give sufficient information on the conference, and on the proposed proceedings. In particular, they should include a list of the expected contributions with their prospective length. Abstracts or early versions (drafts) of the contributions are helpful.

Further remarks and relevant addresses at the back of this book.

Lecture Notes in Mathematics 1526

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Le "**Séminaire de Probabilités**" atteint cette année l'âge inespéré de 25 ans.

La situation des probabilités a bien changé en un quart de siècle. Nous avons essayé de suivre cette évolution alors que les sujets tendaient à se diversifier et le volume des publications à augmenter.

Nous n'avons ni le désir, ni les moyens, de nous transformer en journal mathématique pourvu d'un vaste comité de rédaction.

Cela nous amène, dès ce volume, mais encore davantage à partir du volume XXVII en préparation, à concentrer à nouveau nos efforts sur nos thèmes traditionnels : calcul et analyse stochastiques, étude du mouvement brownien et sujets connexes.

En outre, nous espérons, comme autrefois, accueillir largement des articles d'exposition.

J.A., P.-A.M., M.Y.

SEMINAIRE DE PROBABILITES XXVI

TABLE DES MATIERES

Note de la Rédaction : Les difficultés de communication avec la Russie nous ont empêchés d'obtenir une version définitive, corrigée de certaines erreurs typographiques, des deux articles de S. Kuznetsov contenus dans ce volume, articles dont la qualité nous semble néanmoins - et c'est aussi l'avis du referee - justifier une publication sans délai supplémentaire.

Stochastic Calculus and the Continuity of Local Times of Lévy Processes

Richard Bass* and Davar Khoshnevisan

1. Introduction. Let Z_t be a one dimensional Lévy process with characteristic function

$$E\exp(iuZ_t) = \exp(-t\psi(u)),$$

where

$$\psi(u) = -iau + \frac{1}{2}\sigma^2 u^2 - \int_{-\infty}^{\infty} (e^{iuz} - 1 - iuz1_{(|z|\le 1)})\nu(dz). \tag{1.1}$$

Here ν satisfies $\int (1 \wedge z^2)\nu(dz) < \infty$.

We are interested in those Lévy processes for which 0 is regular for $\{0\}$ and either $\sigma^2 > 0$ or $\nu(\mathbb{R} - \{0\}) = \infty$. In this case (see [K]) there exists a bounded continuous function g that is a density for the 1-resolvent:

$$\int f(x)g(x-y)dx = E^y \int_0^\infty e^{-t} f(Z_t)dt, \qquad f \ge 0, \quad y \in \mathbb{R}. \tag{1.2}$$

(If $G(x,y)$ is the Green function for Z_t killed at an independent exponential time with parameter 1, the relationship between g and G is given by $g(x) = G(0,x) = G(a, a+x)$ for any $a \in \mathbb{R}$ and $G(x,y) = g(y-x)$.)

For each x,

$$g(x) = \frac{1}{2\pi} \int e^{-iux} \frac{1}{1+\psi(u)} du. \tag{1.3}$$

For each $x, g(x - \cdot)$ is the 1-potential of an additive functional L_t^x that is continuous in t. Moreover, a version of $L_t^x(\omega)$ may be chosen that is jointly measurable in (x,t,ω). See [GK] for details. L_t^x is called the local time of Z_t at x. L_t^x is also a density of occupation time measure: if $f \ge 0$,

$$\int_0^t f(Z_s)ds = \int f(x)L_t^x dx, \text{ a.s..} \tag{1.4}$$

A number of people have studied the question of the continuity of L_t^x in the space variable (see [Bo], [Me], [GK] and [MT]), culminating in the works [B1], [BH], and [B2], where a necessary and sufficient condition for the joint continuity of L_t^x in t and x is given.

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The purpose of this paper is to give a stochastic calculus proof of the following sufficient condition for joint continuity. Let $\varphi : \mathbb{R} \to [0,\infty)$ be defined by

$$\varphi^2(x) = \frac{1}{\pi}\int (1-\cos ux)\mathrm{Re}\frac{1}{1+\psi(u)}du. \tag{1.5}$$

Let $d(a,b) = \varphi(b-a)$ and let $H(u)$ be the logarithm of the smallest number of d-balls of radius less than u that are needed to cover $[-1,1]$. Define

$$F(\delta) = \int_0^\delta (H(u))^{\frac{1}{2}}du. \tag{1.6}$$

Theorem 1.1. (a) *If $F(0+) < \infty$, then L_t^x has a jointly continuous version.*

(b) For each t,

$$\limsup_{\delta\downarrow 0} \sup_{\{a,b:\varphi(a-b)<\delta\}} \sup_{s\le t} \frac{|L_s^a - L_s^b|}{F(\varphi(a-b))} \le 2(\sup_x L_t^x)^{1/2}, \quad \text{a.s.}$$

Theorem 1.1(a) was first proved in [BH], where it was also remarked that the entropy condition was equivalent to one involving the monotone rearrangement of φ. Part (b) was also proved in [BH], with, however, the constant 2 replaced by a larger constant (namely 416). In [B2] it was shown that part (b) holds with the constant 2 under the additional assumption that φ is regularly varying (but not slowly varying) and that the constant 2 is sharp. (The principle result of [B2] was that the condition $F(0+) < \infty$ is necessary as well as sufficient for joint continuity.) Marcus and Rosen [MR] have recently obtained necessary and sufficient conditions for the joint continuity of local times of certain Markov processes. Theorem 1.1 for symmetric Lévy processes is a special case of their results.

In Section 2 we prove Theorem 1.1 assuming that $\operatorname{ess\,sup}_x L_t^x < \infty$, a.s. We establish this latter fact in Section 3.

2. Modulus of continuity. Our proof is modeled after that of [McK]. Let us begin by assuming for this section that $\operatorname{ess\,sup}_x L_t^x < \infty$, a.s. Let R be an exponential variable with parameter 1, independent of Z_t. Since $g(x-\cdot)$ is the 1-potential of L_t^x, we have

$$E^a L_R^b = g(b-a). \tag{2.1}$$

Proposition 2.1. $|g(a)-g(b)|\le \varphi^2(a-b)$.

Proof. Let $T_x=\inf\{t: Z_t=x\}, S=T_a\wedge T_b$. Since L_t^x increases only when Z_t is at x, the strong Markov property at time S yields

$$\begin{aligned}|g(a)-g(b)| &= |E^0 L_R^a - E^0 L_R^b| = |E^0[E^{Z_S} L_R^a - E^{Z_S} L_R^b; S\le R]|\\ &\le E^0|E^{Z_S}L_R^a - E^{Z_S}L_R^b|\\ &= E^0[|E^aL_R^a - E^aL_R^b|; S=T_a] + E^0[|E^bL_R^a - E^bL_R^b|; S=T_b]\\ &= |g(0)-g(b-a)|P^0(S=T_a) + |g(a-b)-g(0)|P^0(S=T_b)\end{aligned}$$

Since $g(x)=E^0L_R^x\le E^xL_R^x = g(0)$, then

$$|g(a)-g(b)|\le 2g(0)-g(b-a)-g(a-b).$$

By (1.3) and (1.5), the right hand side equals $\varphi^2(a-b)$. □

Using (2.1) and the Markov property,

$$M_t^a = g(a-Z_{t\wedge R}) - g(a-Z_0) - L_{t\wedge R}^a \tag{2.2}$$

is a martingale with $M_0=0$. Fix a and b and let $N_t=M_t^a-M_t^b$. Let $L_t^*=\operatorname{ess\,sup}_x L_t^x$.

Proposition 2.2. $\langle N,N\rangle_t\le 2\varphi^2(a-b)L_t^*$

Proof. Let N^c, N^d be the continuous and purely discontinuous parts of N_t, respectively. We first estimate $\langle N^d,N^d\rangle_t$.

Let

$$W(x,z)=[\{g(a-(x+z))-g(a-x)\}-\{g(b-(x+z))-g(b-x)\}]. \tag{2.3}$$

Since L_t^a and L_t^b are both continuous in t, the jumps of N_t are the jumps of $g(a-Z_t)$ $-g(b-Z_t)$. Hence

$$[N^d,N^d]_t=\sum_{s\le t}\Delta N_s^2 = \sum_{s\le t\wedge R}(W(Z_{s-},\Delta Z_s))^2.$$

By the definition of Lévy measure, $E\sum_{s\le t}1_A(\Delta Z_s)=\nu(A)t$ if A is a subset of $\mathbb{R}$ that is a positive distance from 0. By the Markov property and the translation invariance of the increments of Z_t, $\sum_{s\le t}1_A(\Delta Z_s)-\nu(A)t$ is a martingale. Taking the stochastic integral of $1_B(Z_{s-})$ with respect to this martingale, we see that $\sum_{s\le t}h(Z_{s-},\Delta Z_s)-$

$\int_0^t \int h(Z_{s-}, z)\nu(dz)\,ds$ is again a martingale, where $h(x,z) = 1_B(x)1_A(z)$. Taking linear combinations and limits, we deduce that

$$[N^d, N^d]_{t\wedge R} - \int_0^{t\wedge R} \int W(Z_{s-}, z)^2 \nu(dz)\,ds$$

is a local martingale. Hence it follows that

$$\langle N^d, N^d \rangle_t = \int_0^{t\wedge R} \int W(Z_{s-}, z))^2 \nu(dz) ds.$$

Since Z_t has only countably many jumps, we get

$$\begin{aligned}
(2.4) \qquad \langle N^d, N^d \rangle_t &= \int_0^{t\wedge R} \int (W(Z_s, z))^2 \nu(dz) ds \\
&\leq \int \int (W(x,z))^2 L_t^x \, dx\, \nu(dz) \\
&\leq L_t^* \int \int (W(x,z))^2 \, dx\, \nu(dz) \\
&= \frac{L_t^*}{2\pi} \int \int |\widehat{W}(u,z)|^2 \, du\, \nu(dz) \qquad \text{(Plancherel's theorem)}
\end{aligned}$$

where $\widehat{W}(u,z)$ is the Fourier transform of $W(\cdot, z)$, z fixed.

By (2.3),

$$\begin{aligned}
\widehat{W}(u,z) &= \widehat{g}(-u)(\{e^{iu(a-z)} - e^{iua}\} - \{e^{iu(b-z)} - e^{iub}\}) \\
&= \widehat{g}(-u)e^{iua}(e^{-iuz} - 1)(1 - e^{iu(b-a)}).
\end{aligned}$$

Since $|e^{iu\theta} - 1|^2 = 2(1 - \cos\theta)$,

$$\begin{aligned}
(2.5) \quad \int \int |\widehat{W}(u,z)|^2 \nu(dz) du &= 2 \int |\widehat{g}(-u)|^2 |1 - e^{iu(b-a)}|^2 \int (1 - \cos uz)\nu(dz)\, du \\
&= 4 \int (1 - \cos(u(b-a)))|\widehat{g}(u)|^2 \mathrm{Re}\, \psi^d(u)\, du,
\end{aligned}$$

where $\psi^d(u) = \psi(u) - \frac{1}{2}\sigma^2 u^2$. Substituting (2.5) in (2.4), we obtain

$$(2.6) \qquad \langle N^d, N^d \rangle_t \leq \frac{2L_t^*}{\pi} \int (1 - \cos(u(b-a)))|\widehat{g}(u)|^2 \mathrm{Re}\, \psi^d(u)\, du.$$

Next we estimate $\langle N^c, N^c \rangle_t$. If f is a smooth function and we write K_t for the martingale part of $f(Z_{t\wedge R})$, then by Itô's formula,

$$K_t^c = \int_0^{t\wedge R} f'(Z_{s-})\sigma dB_s,$$

where B_t is a standard Brownian motion. Then

$$\begin{aligned}
(2.7) \qquad \langle K^c, K^c\rangle_t &= \sigma^2 \int_0^{t\wedge R} (f'(Z_{s-}))^2 ds = \sigma^2 \int_0^{t\wedge R} (f'(Z_s))^2 ds \\
&\le \sigma^2 \int (f'(x))^2 L_t^x dx \\
&\le \sigma^2 L_t^* \int (f'(x))^2 dx \\
&= \sigma^2 L_t^* \frac{1}{2\pi} \int |\widehat{f'}(u)|^2 du \qquad \text{(Plancherel)} \\
&= \sigma^2 L_t^* \frac{1}{2\pi} \int |u|^2 |\hat{f}(u)|^2 du.
\end{aligned}$$

Approximating $g_{ab}(\cdot) = g(a-\cdot) - g(b-\cdot)$ by smooth functions in a suitable way, taking limits, and noting that $\widehat{g}_{ab}(u) = \widehat{g}(-u)(e^{iua} - e^{iub})$, we get

$$\begin{aligned}
\langle N^c, N^c\rangle_t &\le \sigma^2 \frac{L_t^*}{2\pi} \int u^2 |\widehat{g}(-u)|^2 |e^{iua} - e^{iub}|^2 du \\
&= \frac{2L_t^*}{\pi} \int \frac{\sigma^2 u^2}{2} |\widehat{g}(u)|^2 (1 - \cos(u(b-a)))du.
\end{aligned}$$

Adding to (2.6) yields

$$\begin{aligned}
(2.8) \qquad \langle N, N\rangle_t &= \langle N^c, N^c\rangle_t + \langle N^d, N^d\rangle_t \\
&\le \frac{2L_t^*}{\pi} \int |\widehat{g}(u)|^2 \mathrm{Re}\,\psi(u)(1 - \cos(u(b-a)))du.
\end{aligned}$$

Finally, from (1.1), $\mathrm{Re}\,\psi(u) \ge 0$. So

$$\begin{aligned}
(2.9) \qquad \varphi^2(x) &= \frac{1}{\pi} \int (1 - \cos ux) \mathrm{Re} \frac{1}{1+\psi(u)} du \\
&= \frac{1}{\pi} \int (1 - \cos ux) \frac{\mathrm{Re}(1 + \overline{\psi(u)})}{|1+\psi(u)|^2} du \\
&= \frac{1}{\pi} \int (1 - \cos ux) |\widehat{g}(u)|^2 (1 + \mathrm{Re}\,\psi(u))du,
\end{aligned}$$

since $\widehat{g}(u) = (1+\psi(u))^{-1}$. Comparing (2.9) to (2.8) proves the proposition. □

Proposition 2.3. *Let $\epsilon > 0$. There exists $J_0 > 0$ depending on ϵ such that if X_t is any square integrable martingale with jumps bounded in absolute value by J_0 and with $\langle X, X\rangle_t$ continuous, then $\exp(X_t - (1+\epsilon)\langle X, X\rangle_t/2)$ is a positive supermartingale.*

Proof. Take J_0 small enough so that $|e^x - 1 - x| \le (1+\epsilon)x^2/2$ if $|x| \le J_0$. Let

$$Y_t = X_t - (1+\epsilon)\langle X, X\rangle_t/2.$$

By Itô's formula,

$$e^{Y_t} = 1 + \int_0^t e^{Y_{s-}}\,dY_s + \frac{1}{2}\int_0^t e^{Y_{s-}}\,d\langle Y^c, Y^c\rangle_s + \sum_{s\le t}(e^{Y_s} - e^{Y_{s-}} - e^{Y_{s-}}\Delta Y_s)$$

$$= 1 + \int_0^t e^{Y_{s-}}\,dX_s - \frac{(1+\epsilon)}{2}\int_0^t e^{Y_{s-}}\,d\langle X, X\rangle_s + \frac{1}{2}\int_0^t e^{Y_{s-}}\,d\langle X^c, X^c\rangle_s$$
$$+ \sum_{s\le t} e^{Y_{s-}}(e^{\Delta Y_s} - 1 - \Delta Y_s)$$

$$= 1 + \text{ local martingale } - \frac{\epsilon}{2}\int_0^t e^{Y_{s-}}\,d\langle X^c, X^c\rangle_s - \frac{(1+\epsilon)}{2}\int_0^t e^{Y_{s-}}\,d\langle X^d, X^d\rangle_s$$
$$+ \sum_{s\le t} e^{Y_{s-}}(e^{\Delta X_s} - 1 - \Delta X_s)$$

Since $\langle X^d, X^d\rangle_t - \sum_{s\le t}(\Delta X_s)^2$ is a local martingale,

$$\text{(2.10)}\quad e^{Y_t} = 1 + \text{ local martingale } - \frac{\epsilon}{2}\int_0^t e^{Y_{s-}}\,d\langle X^c, X^c\rangle_s + \text{ local martingale}$$
$$- \frac{1+\epsilon}{2}\sum_{s\le t} e^{Y_{s-}}(\Delta X_s)^2 + \sum_{s\le t} e^{Y_{s-}}(e^{\Delta X_s} - 1 - \Delta X_s).$$

But $e^{\Delta X_s} - 1 - \Delta X_s - (1+\epsilon)(\Delta X_s)^2/2 \le 0$ by our selection of J_0. Hence (2.10) exhibits $\exp(Y_t)$ as a local martingale minus an increasing process. □

Write P for P^0.

Corollary 2.4. $P(\sup_{s\le t}|X_s| > \lambda + (1+\epsilon)\langle X, X\rangle_t/2) \le 2e^{-\lambda}$.

Proof. Reducing the continuous part of X_t by stopping times, we may assume X_t bounded, as long as our probability bound does not depend on the L^∞ norm of X_t. We can then write $e^{Y_t} = K_t - V_t$, where K_t is a martingale with $K_0 \equiv 1$ and V_t an increasing process with $V_0 \equiv 0$. Then by Doob's inequality,

$$P(\sup_{s\le t} e^{Y_s} > e^{\lambda}) \le P(\sup_{s\le t} K_s > e^{\lambda})$$
$$\le e^{-\lambda}EK_t = e^{-\lambda}EK_0 = e^{-\lambda}.$$

This proves $P(\sup_{s\le t} X_s > \lambda + (1+\epsilon)\langle X, X\rangle_t/2) \le \exp(-\lambda)$. Applying the same argument to $-X$ proves the corollary. □

Under the assumption $L_t^* < \infty$, a.s., we can now prove Theorem 1.1.

Proof of Theorem 1.1. Let $N_t = M_t^a - M_t^b$ as above, $F(\delta)$ defined by (1.5). Since the potentials of $L_{t\wedge R}^a$ and $L_{t\wedge R}^b$ are bounded. N_t is square integrable ([DM], p.193).

Clearly $F(\delta) \to 0$ as $\delta \to 0$. Also, $\varphi(\delta) \to 0$ as $\delta \to 0$ by the continuity of g, hence $H(u) \to \infty$ as $u \to 0$, hence $\delta/F(\delta) \to 0$ as $\delta \to 0$.

Let α, β be > 0 such that $\alpha\beta > 1$, let $\epsilon > 0$, set $\delta = |b-a|$, and set $\eta = \varphi(\delta)$. Let

$$X_t = \beta F(\eta)\eta^{-2} N_t.$$

Since the jumps of N_t are bounded by $2\sup_x |g(x-a) - g(x-b)| \le 2\varphi^2(b-a)$, the jumps of X_t are bounded by $2\beta F(\eta)$, which will be less than the J_0 of Proposition 2.3 if δ is small.

Now apply Corollary 2.4: if δ is sufficiently small,

$$\begin{aligned}(2.11)\quad & P(\sup_{s\le t}|M_s^a - M_s^b| > \alpha F(\eta) + (1+\epsilon)\beta F(\eta) L_t^*) \\ &\le P(\sup_{s\le t}|N_s| > \alpha F(\eta) + \frac{(1+\epsilon)}{2}\beta\frac{F(\eta)}{\eta^2}\langle N, N\rangle_t) \qquad \text{(Proposition 2.2)} \\ &= P(\sup_{s\le t}|X_s| > \alpha\beta F^2(\eta)/\eta^2 + \frac{1+\epsilon}{2}\langle X, X\rangle_t) \\ &\le \exp(-\alpha\beta F^2(\eta)/\eta^2).\end{aligned}$$

A standard metric entropy argument (see, e.g., [D]) and (2.11) shows that we can find a version of M_t^x that is jointly continuous in $t \in [0, R)$ and $x \in \mathbb{Q}$ and such that for each $K > 0$,

$$(2.12)\qquad P(\limsup_{\eta\downarrow 0}\ \sup_{\{a,b\in\mathbb{Q}\cap[-K,K]:\varphi(a-b)<\eta\}}\ \sup_{s\le t}\frac{|M_s^a - M_s^b|}{F(\varphi(a-b))} > c(\alpha + (1+\epsilon)\beta L_t^*) = 0$$

for each α, $\beta > 0$ such that $\alpha\beta > 1$. Here $\mathbb{Q}$ denotes the rationals. By being a bit more careful with the constants in the metric entropy argument, one can show that one can in fact take $c = 1$.

Fix an ω not in the null set for any α, β, ϵ rational, K a positive integer, take $K \ge \sup_{s\le t}(|Z_s| + 1)$, $\alpha \in [(L_t^*(\omega))^{1/2}, (1+\epsilon)(L_t^*(\omega))^{1/2}]$, and $\beta = (1+\epsilon)/\alpha$, and then let $\epsilon \to 0$. We thus get

$$(2.13)\qquad \limsup_{\eta\downarrow 0}\ \sup_{\{a,b\in\mathbb{Q}:\varphi(a-b)<\eta\}}\ \sup_{s\le t}\frac{|M_s^a - M_s^b|}{F(\varphi(a-b))} \le 2(L_t^*)^{1/2}, \quad \text{a.s.}$$

By Proposition 2.1, $|g(x-a) - g(x-b)| \le \varphi^2(\delta)$. Since $\eta = o(F(\eta))$ as $\eta \to 0$, (2.2) yields (2.13) with $M_s^a - M_s^b$ replaced by $L_{s\wedge R}^a - L_{s\wedge R}^b$. Arguing as in [GK], one can