Lecture Notes in Mathematics

Edited by A. Dold, B. Eckmann and F. Takens

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H. Fujita T. Ikebe S.T. Kuroda (Eds.)

Functional-Analytic Methods for Partial Differential Equations

Proceedings, Tokyo 1989

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Functional-Analytic Methods for Partial Differential Equations

Proceedings of a Conference and a Symposium held in Tokyo, Japan, July 3–9, 1989

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Preface

In commemoration of his retirement from the University of California, Berkeley, an "International Conference on Functional Analysis and its Application in Honor of Professor Tosio Kato" was held on July 3 through 6, 1989, at Sanjo Conference Hall, University of Tokyo, the university where he began his academic career. The Organizing Committee, which consisted of Hiroshi Fujita (Meiji Univ.), S. T. Kuroda (Gakushuin Univ.), and Teruo Ikebe (Kyoto Univ., chairman), selected invited speakers mostly from among his students, students' students, and some recent collaborators. The Conference was followed by a "Symposium on Spectral and Scattering Theory" held on July 7 through 9 at Gakushuin Centennial Memorial Hall, Gakushuin University, Tokyo.

The Conference received financial supports from the Inoue Foundation for Science and the Japan Association for Mathematical Sciences, and the Symposium from Gakushuin University. We express our gratitude to these organizations.

Speakers and participants of these Conference and Symposium wish to heartily dedicate this volume to Professor Kato in celebration of his seventieth birthday.

H. Fujita
T. Ikebe
S. T. Kuroda

Programmes[1]

International Conference on Functional Analysis and its Application in Honor of Professor Tosio Kato

Monday, July 3, 1989

James S. Howland (Univ. of Virginia)
    Quantum Stability
Peter Hess (Univ. of Zürich)
    Singular Perturbations in Periodic-Parabolic Problems
Kenji Yajima (Univ. of Tokyo)
    Smoothing Property of Schrödinger Propagators
Frank. J. Massey III (Univ. of Michigan-Dearborn)
Greg Bachelis (Wayne State Univ.)
    An Optimal Coin Tossing Problem of R. Rivest

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Tosio Kato (Univ. of California, Berkeley)
    Liapunov Functions and Monotonicity for the Navier-Stokes Equation
Hiroshi Matano (Univ. of Tokyo)
    Behavior of Solutions to Elliptic Problems from the Point of View of Dynamical Systems
H. Bruce Stewart (Brookhaven National Lab.)
    Chaos, Bifurcation, and Catastrophe
    After the talk a computer-generated movie "The Lorenz System" completed by H. B. Stewart in 1984 was shown.

Conference Banquet in honor of Professor Kato

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Rafael J. Iório, Jr. (Inst. de Mat. Pura e Aplicada)
    KdV and BO in Weighted Sobolev Spaces
Alan McIntosh (Macquarie Univ.)
    The Square Root Problem for Elliptic Operators
Gustavo Ponce (Pennsylvania State Univ.)
    The Cauchy Problem for the Generalized Korteweg-de Vries Equations
Akira Iwatsuka (Kyoto Univ.)
    On Schrödinger Operators with Magnetic Fields

[1] The titles of the papers contained in the present volume are not necessarily the same as those of talks.

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Arne Jensen (Aalborg Univ.)
    Commutators and Schrödinger Operators
Charles S. Lin (Univ. of Illinois at Chicago)
    On Symmetry Groups of Some Differential Equations
Takashi Ichinose (Kanazawa Univ.)
    Feynman Path Integral for the Dirac Equation

Symposium on Spectral and Scattering Theory

Friday, July 7, 1989

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Peter Hess (Univ. of Zürich)
    The Periodic-Parabolic Eigenvalue Problem, with Applications
Rafael J. Iório, Jr. (Inst. de Mat. Pura e Aplicada)
    Adiabatic Switching for Time Dependent Electric Fields

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Tohru Ozawa (Nagoya Univ.)
    Smoothing Effect for the Schrödinger Evolution Equations with Electric Fields

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Tosio Kato (Univ. of California, Berkeley)
    Positive Commutators $i[f(P), g(Q)]$
James S. Howland (Univ. of Virginia)
    Adiabatic Theorem for Dense Point Spectra
Arne Jensen (Aalborg Univ.)
    High Energy Asymptotics for the Total Scattering Phase in Potential Scattering

Sunday, July 9

Yoshio Tsutsumi (Hiroshima Univ.)
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Alan McIntosh (Macquarie Univ.)
    Operator Theory for Quadratic Estimates

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Spectral Concentration for Dense Point Spectrum

James S. Howland[1]

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Abstract. The degree of spectral concentration at an eigenvalue λ_0 embedded in a dense point spectrum is shown to depend on the extent to which λ_0 is approximated by other eigenvalues whose eigenfunctions have appreciable overlap with the eigenvectors of λ_0. The examples considered include rank one perturbations and time-periodic perturbation of Floquet operators of discrete system.

This article is concerned with the perturbation theory of an eigenvalue λ_0 embedded in a dense point spectrum. This occurs, for example, in connection with Anderson localization or with time-periodic perturbations of discrete systems [2,3,8]. The difficulties involved may be illustrated by recalling the results of Simon and Wolff [14], who show that for certain oprators H_0 with dense pure point spectra, a rank one perturbation leads to an operator

$$H(\beta) = H_0 + \beta\langle\cdot,\varphi\rangle\varphi,$$

which is pure point for *almost every* β. This leaves open the possibility of singular continuous spectrum occurring for arbitrarily small β. The situation is reminiscent of the Stark effect, in which an (isolated) eigenvalue λ_0 disappears into an (absolutely) continuous spectrum for (all) small β.

We shall examine the problem from the point of view of spectral concentration, which was originally invented by Titchmarsh [15] to study the Stark effect. We show that the *degree of concentration depends on the extent to which* λ_0 *is approximated by other eigenvalues whose eigenfunctions have appreciable overlap with the eigenvector*

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of λ_0. A similar phenomenon occurs in the adiabatic theorem for dense point spectrum, with regard to the degree to which the actual motion is approximated by the adiabatic motion [1].

In order to treat these problems, we must first note that the classical theory for isolated eigenvalues extends to the non-isolated case, a fact which seems to have been noted first in the literature by Greenlee [6]. We summarize the necessary results in the first section.

We then treat several examples. We first consider rank one perturbations, as discussed by Aronszajn and Donoghue [5], and Simon and Wolff [14], and then generalize to certain compact perturbations, as in [7]. Finally, we discuss the physically interesting case of a time-periodic perturbation of a discrete Hamiltonian, which has been of considerable recent interest [2,3,8].

The author wishes to thank Barry Simon and David Wales for the hospitality of Caltech, where this work was done.

§1 Spectral Concentration for Non-Isolated Eigenvalues.

We shall assume throughout this section that $H_\beta = \int \lambda dE_\beta(\lambda)$, $0 \le \beta \le \beta_0$, is a family of self-adjoint operators on a Hilbert space $\mathcal{H}$, with $H_\beta \to H_0$ in the strong resolvent sense as $\beta \to 0$; and that λ_0 is an eigenvalue of H_0 of *finite multiplicity* m. Let P_0 be the projection onto the kernel of $H_0 - \lambda_0$.

We say that the spectrum of H_β is *concentrated at* λ_0 on a family of Borel sets S_β iff

$$E_\beta[S_\beta] \to P_0 \tag{1.1}$$

strongly as $\beta \to 0$. For $p \ge 0$, we say that H_β is *concentrated to order* p at λ_0 if the Lebesgue measure

$$|S_\beta| = o(\beta^p), \qquad \text{as } \beta \to 0. \tag{1.2}$$

A *pseudoeigenvector for* H_β *of order* p, or more briefly, a p-*pair* is a family φ_β of unit vectors and a real-valued function λ_β such that

$$(H_\beta - \lambda_\beta)\varphi_\beta = o(\beta^p), \qquad \text{as } \beta \to 0. \tag{1.3}$$

An *asymptotic basis of order* p for H_β at λ_0 is a family $\left\{\varphi_\beta^{(j)}, \lambda_\beta^{(j)} : j = 1, \ldots, m\right\}$ of p-pairs, such that $\lambda_\beta^{(j)} \to \lambda_0$ and $\varphi_\beta^{(j)} \to \varphi^{(j)}$, where $\varphi^{(1)}, \ldots, \varphi^{(m)}$ is a basis of $P_0\mathcal{H}$.

There are two main results of [13]. The first is the equivalence of spectral concentration and the existence of p-pairs. The following is proved in [4], [10, p. 473], and [13] for isolated eigenvalues, and in [6] for non-isolated.

1.1 THEOREM. *If H_β has an asymptotic basis of order p at λ_0, then the spectrum of H_β is concentrated at λ_0 to order p.*

The set S_β is taken as the union of m intervals, centered at $\lambda_\beta^{(j)}$, and of width γ_β where $\gamma_\beta = o(\beta^p)$.

PROOF: The proof is exactly the same as that of Theorem 5.2 of [10, p. 473], except that since λ_0 is not isolated, it must be shown at the end that if $Q = I - P_0$, then

(1.4) $$\text{s-}\lim_{\beta\to 0} E_\beta[S_\beta]Q = 0.$$

Let $J_\varepsilon = (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$. For β small, $S_\beta \subset J_\varepsilon$ so that

$$|E_\beta[S_\beta]Qu| \leq |E_\beta[J_\varepsilon]Qu|.$$

In the limit, by [10, Theorem 1.15, p. 432], this gives

$$\overline{\lim}\,|E_\beta[S_\beta]Qu| \leq |E_0[J_\varepsilon]Qu|.$$

As $\varepsilon \to 0$, the right side converges to $|P_0Qu| = 0$. ∎

1.2 Remark. Riddell also proves the converse result [13, p. 384], that if there is concentration to order p, then an asymptotic basis can be found. We will not need this result, since in practice concentration is usually proved by constructing p-pairs.

The second result of [13] is that p-pairs can be constructed by the perturbation method.

Assume that

(1.5) $$H_\beta = H_0 + \beta V,$$

where V is H_0-bounded, which implies strong resolvent convergence. The *reduced resolvent*

(1.6) $$S = (H_0 - \lambda_0)^{-1}Q$$

is a well-defined self-adjoint operator, although it is bounded only if λ_0 is isolated.

1.3 THEOREM. *Assume that for $k = 1, \cdots, p$, the operators*

(1.7) $$X_1X_2\cdots X_kP_0$$

are all bounded, where each X_j is either S or SV. Then H_β has an asymptotic basis of order p at λ_0.

The idea is that $X_1\cdots X_kP$ are exactly the objects needed to be able to solve the perturbation equations out to order p. For multiplicity $m = 1$, these equations are simple, and as shown in [6], Riddell's proof [13, p. 391] works without change, if one remembers that S need not be bounded. For $m > 1$, Riddell's inductive procedure is less transparent but seems to go through as well. The author has given a different proof for the non-isolated case in [9], basing the argument on Nenciu's idea [11] of applying the adiabatic theorem.

By similar methods, one also obtains

1.4 Theorem. *Let*

$$V(\beta)=\sum_{k=1}^{\infty}\beta^k V^{(k)} \tag{1.8}$$

be bounded and analytic for $|\beta|<\beta_0$. *Assume that the operators*

$$X_1X_2\cdots X_pP_0 \tag{1.9}$$

are all bounded, where each X_j *is either* S *or* $SV^{(k)}$ *for some* $k\le p$. *Then*

$$H_\beta=H_0+V(\beta)$$

is concentrated at λ_0 *to order* p.

§2. Applications.

We shall now give some applications of Theorem 1.3. All will be deduced from a simple corollary. As above, we let λ_0 be an eigenvalue of H_0 of finite multiplicity m, and $S=(H_0-\lambda_0)^{-1}Q$ the reduced resolvent.

2.1 Theorem. *Let* $H_\beta=H_0+\beta V$ *where* V *is* H_0*-bounded. If* S^pV *is bounded, then* H_β *is concentrated at* λ_0 *to order* p.

Proof: This follows from Theorem 1.3 because the operators $X_1X_2\cdots X_p$ are products of operators of the form S^kV with $1\le k\le p$. It is necessary to show that boundedness of S^pV implies boundedness of S^kV for $1\le k\le p$ as well. Let $E=E_0(\lambda_0-1,\lambda_0+1)$ and write

$$S^kV=\left(ES^{k-p}\right)(S^pV)+(1-E)S^kV.$$

Both factors of the first term are bounded, while for the second, we have

$$(1-E)S^kV=\lim_{z\to\lambda_0}\left[V(H_0-z)^{-k}(1-E)\right]^*.$$

The quantity in brackets on the right is bounded and norm analytic at $z=\lambda_0$ since V is H_0-bounded. ∎

We shall also need a real variable lemma [1,7,14].

2.2 Lemma. *If* $a_n>0$ *and* $\sum_{n=1}^{\infty}a_n<\infty$ *then for any sequence* λ_n, *and any* $\alpha\ge 1$,

$$\sum_{n=1}^{\infty}a_n^{\alpha}|\lambda-\lambda_n|^{-\alpha}<\infty \tag{2.1}$$

for a.e. λ.

EXAMPLE 1: RANK ONE PERTURBATIONS [7,14]. Let H_0 be pure point, of finite multiplicity, with $H_0 e_n = \lambda_n e_n$, $n \geq 0$, and e_n a complete orthonormal set. Define

$$H_\beta = H_0 + \beta \langle \cdot, \varphi \rangle \varphi \tag{2.2}$$

where $|\varphi|^2 = 1$. Since the perturbation is of rank one, we may assume that H_0 has simple multiplicity. We are, of course, thinking primarily of the case in which the eigenvalue λ_n are dense in some interval.

One has, for a fixed eigenvalue λ_0,

$$S^p V u = \sum_{n \neq 0} (\lambda_n - \lambda_0)^{-p} \langle \varphi, e_n \rangle \langle u, \varphi \rangle e_n, \tag{2.3}$$

so that $S^p V$ is bounded iff

$$\sum_{n \neq 0} (\lambda_n - \lambda_0)^{-2p} |\langle \varphi, e_n \rangle|^2 < \infty. \tag{2.4}$$

Define, for $p \geq 1$, the set

$$N_p = \left\{ \lambda : \sum_{k=n}^{\infty} |\lambda_k - \lambda|^{-2p} |\langle \varphi, e_k \rangle|^2 = \infty, \text{ for every } n \geq 1 \right\}, \tag{2.5}$$

and

$$N_\infty = \bigcup_{p=1}^{\infty} N_p.$$

According to Lemma 2.2, N_p has Lebesgue measure zero if

$$\sum_{n=0}^{\infty} |\langle \varphi, e_n \rangle|^{1/p} < \infty. \tag{2.6}$$

Hence, if $\langle \varphi, e_n \rangle$ decays exponentially, N_∞ has measure zero. By Theorem 2.1, we have

2.3 THEOREM. *If (2.6) holds, then N_p has measure zero. If $\lambda_0 \notin N_p$, then H_β is concentrated at λ_0 to order p.*

For $p = \infty$, this means that H_β is concentrated to order p for every finite p.

The set N_p consists of points which are well approximated by eigenvalues λ_n whose eigenvectors e_n are substantially disturbed by the perturbation $\langle \cdot, \varphi \rangle \varphi$. Thus, the degree of concentration depends on the degree to which λ_0 can be approximated by such λ_n's. Condition (2.6) assures us that N_p *depends only on the tails* $\{\lambda_k : k \geq n\}$ *of the eigenvalue sequence.*

EXAMPLE 2. COMPACT PERTURBATIONS [7]. A natural generalization of preceding example is the following. Let H_0 be as above, and let

$$H_\beta = H_0 + \beta V,$$

where V is self-adjoint and satisfies

$$\sum_{n=0}^{\infty} |Ve_n|^{1/p} < \infty. \tag{2.7}$$

This is a strong condition which even for $p = 1$ implies that V is trace class (cf. [7]). For *any* λ, define

$$S_0(\lambda) = (H_0 - \lambda)^{-1} Q(\lambda), \tag{2.8}$$

where $I - Q(\lambda)$ is the projection onto $ker(H_0 - \lambda)$ (which may be zero). Then

$$S_0(\lambda)^p Vu = \sum_{\lambda_n \neq \lambda} (\lambda_n - \lambda)^{-p} \langle u, Ve_n \rangle e_n,$$

so that

$$|S_0(\lambda)^p Vu|^2 \leq |u|^2 \sum_{\lambda_n \neq \lambda} (\lambda_n - \lambda)^{-2p} |Ve_n|^2$$

which is finite for a.e. λ by (2.7) and Lemma 2.2.

Define

$$N_p = \left\{ \lambda : \sum_{\lambda_n \neq \lambda} (\lambda_n - \lambda)^{-2p} |Ve_n|^2 = \infty \right\}. \tag{2.9}$$

2.4 THEOREM. *If (2.7) holds, then N_p is of mesure zero. If $\lambda_0 \notin N_p$, then H_β is concentrated to order p at λ_0.*

EXAMPLE 3. FLOQUET HAMILTONIANS [2,3,8]. Next, let H_0 be *discrete,* with eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots$ of *simple multiplicity*. Let $H_0 e_n = \lambda_n e_n$, $|e_n|^2 = 1$. Let $V(t)$ be bounded, strongly C^r and 2π-periodic. We consider the time-dependent Hamiltonian

$$H_\beta(t) = H_0 + \beta V(t),$$

or, more precisely, its Floquet Hamiltonian:

$$K_\beta = i\frac{d}{dt} + H_0 + \beta V(t)$$

on $L_2(0, 2\pi) \otimes \mathcal{H}$ with periodic boundary condition $u(2\pi) = u(0)$. For $\beta = 0$, K_0 has pure point spectrum, with eigenvalues

$$\Lambda_{n,k} = n + \lambda_k$$

$(n = 0, \pm 1, \pm 2, \cdots, k = 1, 2, \cdots)$. We shall *assume that all* $\Lambda_{n,k}$ *are of finite multiplicity*, which implies that they are dense, since the spectum of K_0 is periodic.

Such operators have been of considerable recent interest as a problem in *quantum stability*, which may be said to occur when K_β has pure point spectrum. (See [3,8], and especially [2].) Under conditions weaker than those assumed below, the author [8] showed that K_β has no absolutely continuous spectrum.

Since the degree of spectral concentration can be regarded as a *measure of the stability of an eigenvalue under perturbation*, the following result seems of interest in this context.

Define, for intergers $p \geq 1$, and real $\gamma > 0$, the set

$$N(p, \gamma) = \left\{ \mu : \sum_{n,k}{}'(\lambda_k + n - \mu)^{-2p} k^{-2\gamma} = \infty \right\} \tag{2.10}$$

where the prime on the summation means that terms with $\lambda_k + n = \mu$ are omitted. By assumption, these terms are finite in number.

2.5 LEMMA. *$N(p, \gamma)$ has measure zero if $1 \leq p < \gamma$. Hence for any fixed $\delta > 0$, the set*

$$N_\infty = \bigcup_{p=1}^{\infty} N(p, p + \delta)$$

has measure zero.

PROOF: The set $N(p, \gamma)$ is periodic, with period 1, so it suffices to prove that $N(p, \gamma) \cap J$ is of measure zero, where $J = [0, 1)$. Fix $\mu \in (0, 1)$ and write the sum in (2.10) as

$$\left\{ \sum_{\lambda_k + n \in J} + \sum_{\lambda_k + n \notin J} \right\} (\lambda_k + n - \mu)^{-2p} k^{-2\gamma}.$$

The second term is analytic on $(0, 1)$ and thus always finite. The first term can be treated by observing that $\lambda_k + n \in J$ for *at most one value* n_k of n. Thus the term is equal to

$$\sum_k{}'(\lambda_k + n_k - \mu)^{-2p} k^{-2\gamma} \varepsilon_k,$$

where ε_k is zero if $\lambda_k + n$ is never in J, and is one otherwise. By lemma 2.2, this sum is finite for a.e. μ if $p < \gamma$.

2.6 THEOREM. *Assume that K_0 has finite multiplicity, that $V(t)$ is strongly C^∞, and that the gap*

$$\Delta\lambda_n = \lambda_{n+1} - \lambda_n$$

between eigenvalues satisfies

$$\Delta\lambda_n \geq c n^\alpha$$

for some $\alpha > 0$. Then the spectrum of K_β is concentrated at λ_0 to all orders if $\lambda_0 \notin N_\infty$.

PROOF: According to [8], K_β is unitarily equivalent to an operator of the form

$$\tilde{K}_\beta = i\frac{d}{dt} + \tilde{H} + \beta A W(t,\beta) A,$$

where $\tilde{H}$ is discrete and diagonal in the same basis e_n as H, $W(t,\beta)$ is bounded and analytic in β, and

$$A = \sum_n n^{-\gamma} \langle \cdot, e_n \rangle e_n.$$

For $V(t)$ in C^∞, γ may be taken as large as desired.

For p fixed, choose $\gamma > p + \delta$. By Lemma 2.5, $S(\lambda)^p A$ is bounded. Thus if we expand the perturbation

$$V(\beta) = AW(\beta)A = \sum_{k=1}^{\infty} \beta^k A W^{(k)} A,$$

we will have

$$S^p(\lambda) V^{(k)} = (S^p(\lambda)A)(W^k A)$$

bounded for all k. Using Theorm 1.4, and the argument in the proof of Theorem 2.1 gives the result. ∎

REMARK. It is possible to keep track of the relationship between α, the degree of smoothness of $V(t)$, and the order of concentration that can be expected.

§3 Remarks.

There are two points that need clarification. In the first place, *is concentration really relevant here?* For example, in [2,3], a KAM-type argument leads to an explicit diagonalization of H_β, so that the spectrum is concentrated on a *one point* set, the perturbed eigenvalue. Of course, [2,3] contain strong assumptions, like analyticity, but the question is still in order.

A complete answer would require a rather complete theory of these operators, which we are at present far from having. Nevertheless, the following example is instructive.

Let ν be a measure on $[0,1]$ which is singular continuous, for which the set N of λ where

$$\int_0^1 (\lambda - t)^{-2} \nu(dt) = \infty \tag{3.1}$$

is *of measure zero, but dense.* Let the operator H_0 of multiplication by λ on $L^2(\nu)$ be perturbed by the vector 1:

$$H_\beta = H_0 + \beta \langle \cdot, 1 \rangle 1$$

(cf. [14]). Then H_β is pure point for a.e. β, but can have *no point spectrum* in N [8,14]. This means that H_β cannot have an eigenvalue $\lambda(\beta)$ which varies continuously, as in the case [2,3]. The author finds it probable that worse examples can be constructed.

In the second place, *how do we know that all the eigenvalues of H_0 are not in the bad set N_p ?* In this case, our theorems would say nothing! While it might be possible for this to occur, the following result shows that it is in some sense rare.

Recall that if H is a self-adjoint, we write $P(\lambda)$ for the projection onto the kernel of $H - \lambda$ (which may be trivial), $Q(\lambda) = I - P(\lambda)$, and $S(\lambda) = (H - \lambda)^{-1}Q(\lambda)$ for the reduced resolvent. By definition, $\lambda \notin N_p(H, V)$ iff $S(\lambda)^p V$ is bounded.

3.1 THEOREM. *Let $H = H_0 + V$, where V is H_0-bounded, and assume that $\lambda \notin N_p(H_0, V)$ and that $S_0(\lambda)$ is compact. If λ is an eigenvector of H or H_0, we also assume that its multiplicity is finite.*

(a) *if $\lambda \notin \sigma_p(H)$, then $\lambda \notin N_p(H, V)$.*
(b) *if $\lambda \in \sigma_p(H)$, then $\lambda \notin N_{p-1}(H, V)$.*

Hence, in general, $N_{p-1}(H, V) \subset N_p(H_0, V)$.

If we apply this result to Example 1, where

$$H_\beta = H_0 + \beta\langle\cdot, \varphi\rangle\varphi,$$

we see that $N_{p-1}(H_\beta) \subset N_p(H_0)$. According to [8] and [14], however, for *any fixed null set N*

$$E_\beta[N] = 0$$

for a.e. β. Thus if

$$\sum_n |\langle\varphi, e_n\rangle|^{1/p} < \infty,$$

then for a.e.β, $\lambda \notin N_{p-1}(H_\beta)$ *for every eigenvalue of H_β*. This indicates that having eigenvalues in N_p is an unstable condition and does not obtain in the generic case. Thus, here, the perturbation problem

$$H'_\beta = H'_0 + \beta\langle\cdot, \varphi\rangle\varphi$$

with

$$H'_0 = H_0 + \beta_0\langle\cdot, \varphi\rangle\varphi$$

has $N_{p-1}(H'_0) \cap \sigma_p(H'_0) = \emptyset$ for a.e. β_0.

We sketch a proof, leaving some details about domains to the reader.

PROOF OF THEOREM 3.1: First, note that we can assume that $\lambda \notin \sigma_p(H_0)$ by writing

$$H = H_0 + V = (H_0 + P_0) + (V - P_0) = H'_0 + V', \tag{3.2}$$

where $P_0 = P_0(\lambda)$. Then

$$\begin{aligned} R'_0(\lambda)^p V' &= [S_0(\lambda)^p + P_0](V - P_0) \\ &= S_0(\lambda)^p V + P_0 V - P_0 \end{aligned} \tag{3.3}$$

is bounded and compact for $p = 1$. Thus, replacing H_0 and V by H_0' and V', we can assume $\lambda \notin \sigma_p(H_0)$, and hence that $I + R_0(\lambda)V$ has a bounded inverse.

For part (a), observe that (suppressing λ)

$$(3.4) \qquad \begin{aligned} R_0^n - R^n &= \sum_{k=1}^{n} R_0^k R^{n-k} - R_0^{k-1} R^{n-k+1} \\ &= \sum_{k=1}^{n} R_0^{k-1}[R_0 - R]R^{n-k} = \sum_{k=1}^{n} R_0^k V R^{n-k+1} \\ &= R_0 V R^n + \sum_{k=2}^{n} R_0^k V R^{n-k+1}. \end{aligned}$$

Solving for $\mathbf{R}^n$ gives

$$(3.5) \qquad R^n V = [I + R_0 V]^{-1} R_0^n V - \sum_{k=2}^{n} R_0^k V R^{n-k+1} V.$$

It follows by induction that $R_0^p V$ bounded implies $R^p V$ bounded.

For (b), suppose that $\lambda \in \sigma_p(H)$. Then every eigenvector ψ satisfies

$$(3.6) \qquad \psi = -R_0(\lambda) V \psi$$

so that

$$(3.7) \qquad R_0(\lambda)^{p-1}\psi = R_0(\lambda)^p V \psi$$

or, in other words,

$$R_0(\lambda)^{p-1} P(\lambda)$$

is bounded. As above, write

$$H' = H + P = H_0 + (V + P) = H_0 + V'.$$

Then

$$(3.8) \qquad R_0^{p-1} V' = R_0^{p-1} V + R_0^{p-1} P$$

is bounded, while

$$(3.9) \qquad (R')^{p-1} V' = S^{p-1} V + (PV + P).$$

The last two terms are bounded, so S^{p-1} is bounded iff $(R')^{p-1}V'$ is. Applying (a) now yields the result. ∎