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Edited by A. Dold and B. Eckmann

Subseries: Mathematisches Institut der Universität und Max-Planck-Institut für Mathematik, Bonn

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Jürgen Jost

Harmonic Maps Between Surfaces

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Harmonic Maps Between Surfaces

(with a Special Chapter on Conformal Mappings)

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Dedicated to the memory of Dieter Kieven

P R E F A C E

The purpose of these Lecture Notes is twofold. On one hand, I want to give a fairly complete and self-contained account of the results on harmonic maps between surfaces. On the other hand, these notes should also serve as an introduction to the theory of harmonic maps in general; therefore, whenever appropriate, I point out which of the two-dimensional results pertain to higher dimensions and which do not, and I try to give some references and an idea of the respective proof. For a more complete account in this direction, however, the reader should consult the several excellent survey articles of Eells and Lemaire.

An essential aim of this book is to show the variety of methods and the interplay of different fields in the theory of harmonic maps, in particular the calculus of variations, partial differential equations, differential geometry, algebraic topology, and complex analysis. Thus, the concept of this book is strongly opposed to the view of a mere specialist. In particular, I think that a completely unified treatment of the topic is neither possible nor desirable.

Nevertheless, I believe that this treatment contains several simplifications and unifications compared to the presentations available in the existing literature. This book is not intended as a mere enumeration of unrelated results. On the contrary, the sequence of the chapters also reflects a logical order, and many different tools have to be constructed, until the results of the three final chapters can be proved. In particular, conformal mappings are used in a much more thorough way than in the existing literature. An outline of the contents now follows.

After giving an account of the history and presenting the definition of harmonic maps from several points of view in chapter 1, we start in chapter 2 with some geometric considerations. These concern convex discs on surfaces, and the result is roughly that if on a disc there are no conjugate points then there are also no cut points.

Moreover, we show the existence of local coordinates with curvature controlled Christoffel symbols, following Jost-Karcher [JK1] .

Chapter 3 deals with conformal mappings. We first prove Theorem 9.3 in Morrey, "Multiple Integrals...", Springer, 1966, since Morrey's proof contains a mistake. The difficulty which leads to this error is overcome by minimizing energy in a restricted subclass of the Sobolev space H^1_2 which is suitably adapted to the problem, so that we can nevertheless conclude that the minimum is a conformal map with the desired properties (we shall encounter a similar idea in chapters 4 and 11). Furthermore, we shall prove that this map is a global diffeomorphism and as regular as one could expect (A-priori estimates, however, will only be obtained in later chapters).

In chapter 4, we first solve the Dirichlet problem for the case that the boundary values lie in some convex ball, a result due to Hildebrandt-Kaul-Widman [HKW 3]. Our proof consists of a combination of a rather general maximum principle and a lemma due to Courant and Lebesgue which is only valid in two dimensions. The proof also gives a-priori estimates for the modulus of continuity of the harmonic map. We then attack the general existence problem for harmonic maps between compact surfaces. Using the Courant-Lebesgue Lemma again, it is not hard to see that the limit of an energy minimizing can fall out of a homotopy class only if a sphere splits off. If the second homotopy group of the image vanishes, this cannot happen, however, and we thus obtain a new proof of the fundamental existence theorems of Lemaire [L1], [L2]. Furthermore, by a careful replacement argument, we can also solve the Dirichlet problem in two different homotopy classes for nonconstant boundary values, if the image is homeomorphic to a 2-sphere. In chapter 5, we deal with the question of uniqueness of harmonic maps and prove the corresponding results of Hartmann [Ht] and Jäger-Kaul [JäK1] and then examine in some more detail the case of maps between closed surfaces.

In chapter 6, we prove $C^{1,\alpha}$-a-priori-estimates for the case where the domain is the unit disc in the plane. This latter assumption can then be removed with the help of the results of chapter 7 where we prove estimates for the functional determinant from below for univalent harmonic mappings between surfaces. These estimates apply in particular to conformal maps, and since a conformal map composed with a harmonic one is again harmonic, we can use the result of chapter 3 to pass from the unit disc to an arbitrary domain in chapter 9. The results of chapter 6 and 7 are based on Jost-Karcher [JK1] and employ several important ideas of E. Heinz.

In chapter 8, we prove the existence of harmonic diffeomorphisms as solutions of the Dirichlet problem, if the boundary values map the boundary of the domain homeomorphically onto a convex curve inside a convex disc. This result is taken from [J3] and uses in particular the results of chapter 7.

We can also use the a - priori - estimates to provide non - variational proofs of Thms. 4.1 and 8.1 in chapter 9, using Leray - Schauder degree theory.
We then apply Theorem 8.1 in chapter 10 to prove the existence of harmonic coordinates on arbitrary discs on a surface according to Jost - Karcher [JK1]. These coordinates possess best possible regularity properties and can be used to prove $C^{2,\alpha}$ - a - priori estimates for harmonic maps between surfaces depending only on curvature bounds and injectivity radii, once the modulus of continuity is known.
Theorem 8.1 will again be applied in chapter 11 where we prove the existence of harmonic diffeomorphisms between closed surfaces, due to Jost - Schoen [JS]. We minimize energy in the class of diffeomorphisms and then apply a rather delicate replacement argument to show that the limit is a harmonic diffeomorphism.
The final chapter gives some applications of harmonic maps between surfaces. First, we give the analytic proof of Eells - Wood of a well known result of Kneser concerning mappings between closed surfaces, and then we give some applications of Earle - Eells and Tromba of harmonic maps to Teichmüller theory.
Furthermore, we discuss the Theorem of Ruh - Vilms stating that the Gauss map of a submanifold of Euclidean space with constant mean curvature is harmonic,as well as immersed surfaces in 3 - space of constant Gauss curvature.
Among the omissions of the present book are results on the explicit construction and classification of harmonic maps between manifolds with canonical metrics. We refer the reader to [L 1], [EW 2], [EW 3], [EL 3] instead, since we do not feel that we can contribute anything new to the presentation of this area.
My work is indebted to several persons. To Hermann Karcher, I owe many insights into the geometric aspects of the field which he generously communicated to me. Furthermore, I benefitted much from collaboration or conversations with Jim Eells, Bob Gulliver, Luc Lemaire, Rick Schoen (in particular, Chapter 11 represents joint work with him), John Wood and Shing-Tung Yau. But most of all, I am indebted to Stefan Hildebrandt for his continous advice and encouragement over many years, and for supporting my research in every possible way through the means of the Sonderforschungsbereich 72 at the University of Bonn. Finally, I am grateful to Alfred Baldes for some useful comments on my manuscript and to Monika Zimmermann for typing it with great care and patience.

Table of contents

1. Introduction

1.1. A short history of variational principles

Among the first persons to realize the importance of variational problems and the physical significance of their solutions was G. W. Leibniz (1646 - 1716). In his work, however, mathematical and physical reasoning was closely interwoven with philosophical and theological arguments. One of the aims of his philosophy was to solve the problem of theodizee, i.e. to reconcile the evil in the world with God's goodness and almightiness (cf. [Lz]).Leibniz' answer was that God has chosen from the innumerable possible worlds the best possible, but that a perfect world is not possible. (This infinite multitude can only be conceived by an infinite understanding, which provided a proof of the existence of God for Leibniz.) This best possible world is distinguished by a pre - establishe harmony between itself, the kingdom of nature, on one hand and the heavenly kingdom of grace and freedom on the other hand. Through this the effective causes unite with the purposive causes. Thus bodies move due to their own internal laws in accordance with the thoughts and desires of the soul. In this way, the contradiction between the predetermination of the physical world following strict laws and the constantly experienced spontaneity and freedom of the individual is removed. The best possible world must here obey specific laws since an ordered world is better than a chaotic one. This proves therefore the necessity of the existence of natural laws. The contents of the natural laws, however, are not completely determined as is the case for geometric laws but are only determined in a moral sense, since they must satisfy the criteria of beauty and simplicity in the best of all possible worlds. This leads Leibniz even to variational principles. This is because if a physical process did not yield an extreme value, a maximum or minimum, for a particular energy or action integral, the world could be improved and would therefore not be the best possible one. Conversely, Leibniz also uses the beauty and simplicity of natural laws as evidence for his thesis of pre - established harmony. (The notion that we live in the best possible world was frequently rejected and even ridiculed by subsequent critics, in particular Voltaire, on account of the apparent flaws of this world, but Leibniz' point that a perfectly good world is not possible was beyond reach of these arguments.)

Leibniz, however, did not elaborate his argument concerning variational principles in his publications, but only in a private letter. Thus, it happened that a principle of least (and not only stationary) action was later rediscovered by Maupertuis (1698 - 1759), without knowing of Leibniz' idea. When S. König (1712 - 1757) then claimed priority for Leibniz on account of his letter that he was not able to show however to the Prussian Academy of Sciences (whose president was Maupertuis) this led to one of the most famous priority controversies in scientific history in which even Voltaire, Euler, and Frederick the Great became involved. It was also pointed out that Maupertuis' principle of least action should be replaced by a principle of stationary action since physical equilibria need only be stationary points but not necessarily minima of variational problems.

1.2. The concept of geodesics

One of the variational problems of most physical importance and mathematical interest was the problem of geodesics, i.e. to find the shortest (or at least locally shortest) connections between two points in a metric continuum, e.g. a Riemannian manifold. Applications of this concept range from general relativity, where geodesics describe the paths of moving bodies , to many innermathematical applications.
Geodesics are critical points of the length integral

$$\int_0^1 \left|\frac{\partial}{\partial t} c\right| dt$$

where $c:[0,1] \to N$ is the parametrization, as well as, if they are parametrized proportionally to arclength, of the energy integral

$$\int_0^1 \left|\frac{\partial}{\partial t} c\right|^2 dt.$$

Here, unfortunately, we find some ambiguity of terminology, since the mathematical term "energy" corresponds to the physical concept of "action", while in physics "energy" has a different meaning.
Because of the many applications of geodesics, it was rather natural to generalize this concept. While minimal surfaces are critical points of a twodimensional analogue of the length integral, namely the area integral, the generalization of the energy integral for maps between Riemannian manifolds led to the concept of harmonic maps. They are critical points of the corresponding integral where the squared norm of the gradient or energy density has to be defined in terms intrinsic to the geometry of the domain and target manifold and the map between them.

1.3. Definition of harmonic maps

Suppose that X and Y are Riemannian manifolds of dimensions n and N , resp., with metric tensors $(\gamma_{\alpha\beta})$ and (g_{ij}), resp., in some local coordinate charts $x = (x^1,\dots,x^n)$ and $f = (f^1,\dots,f^N)$ on X and Y , resp. Let $(\gamma^{\alpha\beta}) = (\gamma_{\alpha\beta})^{-1}$. If $f: X \to Y$ is a C^1 - map, we can define the energy density

$$e(f) := \frac{1}{2}\gamma^{\alpha\beta}(x)\, g_{ij}(f)\, \frac{\partial f^i}{\partial x^\alpha}\frac{\partial f^j}{\partial x^\beta}$$

where we use the standard summation convention (greek minuscules occuring twice are summed from 1 to n , while latin ones are summed from 1 to N) and express everything in terms of local coordinates. Then the energy of f is simply

$$E(f) = \int_X e(f)\,dX$$

If f is of class C^2 and $E(f) < \infty$ and f is a critical point of E[1)], then f is called harmonic and satisfies the corresponding Euler - Lagrange -equations. These are of the form

$$\text{(1.3.1)} \qquad \frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{\gamma}\,\gamma^{\alpha\beta}\frac{\partial}{\partial x^\beta} f^i\right) + \gamma^{\alpha\beta}\Gamma^i_{jk}\frac{\partial}{\partial x^\alpha} f^j \frac{\partial}{\partial x^\beta} f^k = 0$$

in local coordinates, where $\gamma = \det(\gamma_{\alpha\beta})$ and the Γ^i_{jk} are the Christoffel symbols of the second kind on Y .

We thus obtain a nonlinear elliptic system of partial differential equations, where the principal part is the Laplace - Beltrami operator on X and is therefore in divergence form, while the nonlinearity is quadratic in the gradient of the solution.

We now want to look at the definition of harmonic maps from a more intrinsic point of view. The differential df of f , given in local coordinates by

$$df = \frac{\partial f^i}{\partial x^\alpha}\, dx^\alpha\, \frac{\partial}{\partial f^i}$$

can be considered as a section of the bundle $T^*X \otimes f^{-1}TY$. Then

$$e(f) = \frac{1}{2}\gamma^{\alpha\beta} \left< \frac{\partial f}{\partial x^\alpha}, \frac{\partial f}{\partial x^\beta} \right>_{f^{-1}TY}$$

$$= \frac{1}{2} \left< df, df \right>_{T^*X\otimes f^{-1}TY}$$

1) w.r.t. variations vanishing on ∂X , in case $\partial X \neq \emptyset$.

i.e. $e(f)$ is the trace of the pullback via f of the metric tensor of Y. In particular, $e(f)$ and hence also $E(f)$ are independant of the choice of local coordinates and thus intrinsically defined.
f is harmonic, if

(1.3.2) $\tau(f) = 0$,

where $\tau(f) = \text{trace}\, \nabla df$, and ∇ here denotes the covariant derivative in the bundle $T^*X \otimes f^{-1}TY$.
Let us quickly show, why (1.3.1) and (1.3.2) are equivalent (cf. [EL 4]).

$$\nabla_{\frac{\partial}{\partial x^\beta}} (df) = \nabla_{\frac{\partial}{\partial x^\beta}} \left(\frac{\partial f^i}{\partial x^\alpha} dx^\alpha \frac{\partial}{\partial f^i}\right)$$

$$= \frac{\partial}{\partial x^\beta}\left(\frac{\partial f^i}{\partial x^\alpha}\right) dx^\alpha \frac{\partial}{\partial f^i} + \left(\nabla^{T^*X}_{\frac{\partial}{\partial x^\beta}} dx^\alpha\right) \frac{\partial f^i}{\partial x^\alpha}\frac{\partial}{\partial f^i} +$$

$$+ \left(\nabla^{f^{-1}TY}_{\frac{\partial}{\partial x^\beta}} \frac{\partial}{\partial f^i}\right) \frac{\partial f^i}{\partial x^\alpha} dx^\alpha =$$

$$= \frac{\partial^2 f^i}{\partial x^\alpha \partial x^\beta} dx^\alpha \frac{\partial}{\partial f^i} - {}^X\Gamma^\alpha_{\beta\gamma} dx^\gamma \frac{\partial f^i}{\partial x^\alpha}\frac{\partial}{\partial f^i} + {}^Y\Gamma^k_{ij} \frac{\partial}{\partial f^k}\frac{\partial f^j}{\partial x^\beta}\frac{\partial f^i}{\partial x^\alpha} dx^\alpha \quad {}^{1)}$$

and thus, since $\tau(f) = \text{trace}\, \nabla df$,

$$\tau^k(f) = \gamma^{\alpha\beta} \frac{\partial^2 f^k}{\partial x^\alpha \partial x^\beta} - \gamma^{\alpha\beta}\, {}^X\Gamma^\gamma_{\alpha\beta} \frac{\partial f^k}{\partial x^\gamma} + \gamma^{\alpha\beta}\, {}^Y\Gamma^k_{ij} \frac{\partial f^i}{\partial x^\alpha}\frac{\partial f^j}{\partial x^\beta} ,$$

and we see that (1.3.1) and (1.3.2) are equivalent.
From the preceding calculation, we see that the Laplace - Beltrami operator is the contribution of the connection in T^*X , while the connection in $f^{-1}TY$ gives rise to the nonlinear term involving the Christoffel symbols of the image.
With the preceding notations, we can also calculate the Hessian of a harmonic map f for vector fields v , w along f (i.e. v and w are sections of $f^{-1}TY$). For this purpose, we consider a two - parameter variation f_{st} with

$$v = \frac{\partial f_{st}}{\partial s}\Big|_{s,t=0} , \quad w = \frac{\partial f_{st}}{\partial t}\Big|_{s,t=0}$$

1) Here, we distinguish the Christoffel symbpls of X and Y by the superscript X or Y , resp.

We then want to calculate

$$H_f(v,w) := \frac{\partial^2 E(f_{st})}{\partial s \partial t} \Big| \; s,t = 0$$

We have, writing f instead of f_{st}, and taking scalar products $\langle \cdot , \cdot \rangle$ in $T^*X \otimes f^{-1}TY$, if not otherwise indicated,

$$\frac{\partial}{\partial t} \frac{\partial}{\partial s} \frac{1}{2} \left\langle \frac{\partial f}{\partial x^\alpha} dx^\alpha , \frac{\partial f}{\partial x^\beta} dx^\beta \right\rangle =$$

$$= \frac{\partial}{\partial t} \left\langle \nabla_{\frac{\partial}{\partial s}} \frac{\partial f}{\partial x^\alpha} dx^\alpha , \frac{\partial f}{\partial x^\beta} dx^\beta \right\rangle =$$

$$= \frac{\partial}{\partial t} \left\langle \nabla^{f^{-1}TY}_{\frac{\partial}{\partial x^\alpha}} \left(\frac{\partial f}{\partial s}\right) dx^\alpha , \frac{\partial f}{\partial x^\beta} dx^\beta \right\rangle =$$

$$= \left\langle \nabla_{\frac{\partial}{\partial t}} \nabla^{f^{-1}TY}_{\frac{\partial}{\partial x^\alpha}} \left(\frac{\partial f}{\partial s}\right) dx^\alpha , \frac{\partial f}{\partial x^\beta} dx^\beta \right\rangle +$$

$$+ \left\langle \nabla^{f^{-1}TY}_{\frac{\partial}{\partial x^\alpha}} \left(\frac{\partial f}{\partial s}\right) dx^\alpha , \nabla^{f^{-1}}_{\frac{\partial}{\partial x^\beta}} \left(\frac{\partial f}{\partial t}\right) dx^\beta \right\rangle =$$

$$= \left\langle \nabla^{f^{-1}TY}_{\frac{\partial}{\partial x^\alpha}} \nabla_{\frac{\partial}{\partial t}} \left(\frac{\partial f}{\partial s}\right) dx^\alpha , \frac{\partial f}{\partial x^\beta} dx^\beta \right\rangle$$

$$+ \left\langle R^N \left(\frac{\partial f}{\partial x^\alpha} dx^\alpha , \frac{\partial f}{\partial t}\right) \frac{\partial f}{\partial s} , \frac{\partial f}{\partial x^\beta} dx^\beta \right\rangle$$

$$+ \left\langle \nabla^{f^{-1}TY}_{\frac{\partial}{\partial x^\alpha}} v \, dx^\alpha , \nabla^{f^{-1}TY}_{\frac{\partial}{\partial x^\beta}} w \, dx^\beta \right\rangle ,$$

Now

$$\int_X \left\langle \nabla^{f^{-1}TY}_{\frac{\partial}{\partial x^\alpha}} \nabla_{\frac{\partial}{\partial t}} \frac{\partial f}{\partial s} dx^\alpha , \frac{\partial f}{\partial x^\beta} dx^\beta \right\rangle dX$$

$$= \int \frac{\partial}{\partial x^\alpha} \left(\gamma^{\alpha\beta} \left\langle \nabla_{\frac{\partial}{\partial t}} \frac{\partial f}{\partial s} , \frac{\partial f}{\partial x^\beta} \right\rangle_{f^{-1}TY} \sqrt{\gamma}\right) dx^1 \ldots dx^n$$

$$+ \int \left\langle \nabla_{\frac{\partial}{\partial t}} \frac{\partial f}{\partial s} dx^{\alpha} , \nabla_{\frac{\partial}{\partial x^{\alpha}}} \frac{\partial f}{\partial x^{\beta}} dx^{\beta} \right\rangle$$

$$= \int \left\langle \nabla_{\frac{\partial}{\partial t}} \frac{\partial f}{\partial s} , \gamma^{\alpha\beta} \nabla_{\frac{\partial}{\partial x^{\alpha}}} \frac{\partial f}{\partial x^{\beta}} \right\rangle_{f^{-1}TY}$$

by Stokes' Theorem

$= 0$, since $\gamma^{\alpha\beta} \nabla_{\frac{\partial}{\partial x^{\alpha}}} \frac{\partial f}{\partial x^{\beta}} = \text{trace} \nabla df = 0$, as f is harmonic.

Thus

$$H_f(v,w) = \int_X \gamma^{\alpha\beta} \left\langle \nabla^{f^{-1}TY}_{\frac{\partial}{\partial x^{\alpha}}} v , \nabla^{f^{-1}TY}_{\frac{\partial}{\partial x^{\beta}}} w \right\rangle_{f^{-1}TY}$$

$$- \int_X \gamma^{\alpha\beta} \left\langle R^N\left(\frac{\partial f}{\partial x^{\alpha}} , v\right) \frac{\partial f}{\partial x^{\beta}} , w \right\rangle_{f^{-1}TY}$$

$$= \int_X \left\langle \nabla^{f^{-1}TY} v , \nabla^{f^{-1}TY} w \right\rangle_{f^{-1}TY}$$

$$- \int_X \text{trace}_X \left\langle R^N(df,v)\, df , w \right\rangle_{f^{-1}TY}$$

For the preceding calculations cf. also [EL 4] .
We now want to look at the definition of harmonic maps from a somewhat different point of view. By the famous embedding theorem of Nash ([Na]), Y can be isometrically embedded in some Euclidean space $\mathbb{R}^1$.
We define the Sobolev space

$$W_2^1(X,Y) = \{f \in W_2^1(X,\mathbb{R}^1) \ : \ f(x) \in Y \ \text{ a.e.}\}$$

Since $W_2^1(X,\mathbb{R}^1) = H_2^1(X,\mathbb{R}^1)$ by a well-known theorem of Meyers and Serrin