Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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Proceedings, Mysore 1981

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ORGANIZER'S FOREWORD

MATSCIENCE, The Institute of Mathematical Sciences, Madras, conducts three conferences a year, one in pure mathematics, one in theoretical physics and another on applications of mathematical methods in the garden city of Mysore or the hill resort of Ootacamund. Since 1979, in pure mathematics, in conformity with the Ramanujan tradition, it was decided to conduct annual conferences in Number Theory with the active cooperation of the number theory group at the Tata Institute of Fundamental Research, Bombay. It was our good fortune to have had Professor Paul Erdös as the distinguished invitee for our Third Conference on Number Theory held at Mysore during June 3-6, 1981 with Professors K.Ramachandra of the Tata Institute of Fundamental Research, R.P.Bambah of the Punjab University, Chandigarh and Krishnaswami Alladi, till recently at the University of Michigan and now at MATSCIENCE, as the other principal lecturers.

According to the established conventions of our Institute the number of participants is usually limited to about thirty to provide for one hour lectures and enough time for discussions. These proceedings comprise ten of the invited addresses and contributed papers besides problems proposed in a special session chaired by Professor Erdös.

We are grateful to all the participants for their active and enthusiastic cooperation and in particular to Professor Paul Erdös who flew to Madras directly from Waterloo, Canada immediately after receiving an honorary doctorate there, to be in time for MATSCIENCE Conference.

It is indeed an example of international collaboration that the Springer Verlag, through its Editor Professor Dold, is publishing these proceedings in their lecture notes series. The contributions not included in this volume are being published elsewhere. A complete list of participants and their addresses are given overleaf.

ALLADI RAMAKRISHNAN

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Editor's Preface

Inspired by the work of the legendary Indian mathematician Srinivasa Ramanujan, there has been a lot of activity in Number Theory in India during this century, especially in the elementary and analytic aspects of the subject. However, few conferences conducted in India have been devoted only to Number Theory. The Matscience Conferences on Number Theory are therefore the only series of their kind, and in this third conference gained momentum due to the active participation of Professor Paul Erdös. Many of India's number theorists presented papers in the following areas: elementary number theory, analytic number theory, probabilistic number theory, the geometry of numbers and number theoretic questions with a special combinatorial appeal. As perhaps the greatest problem proposer in recent decades, it was only fitting that Professor Erdös chaired a problem session. Although not all papers or problems presented at the conference have appeared in these proceedings, what has been assembled here is fairly representative of the theme of the conference. As one of the participants, I thank 'Matscience' for their generous financial support which enabled the conference to fulfil its aims and objectives.

Krishnaswami Alladi
Princeton, New Jersey
April 1982

CONTENTS

ADDITIVE FUNCTIONS AND SPECIAL SETS OF INTEGERS

Krishnaswami Alladi

Abstract

An additive function f is one that satisfies $f(m.n) = f(m) + f(n)$ for positive intègers m and n with $(m,n) = 1$. A special set of integers S is any subset of the positive integers. We discuss the value distribution of $f(n)$ for $n \in S$. Generalisations and extensions of the Hardy-Ramanujan results on normal order, the Turán-Kubilius inequality, and the Erdös-Kac theorem are obtained. Two kinds of special sets are considered. One class consists of sets S that are obtained as multiplicative semigroups generated by a prescribed collection of prime powers. The second class of sets are those in which the frequency of elements which are multiples of an integer d can be given in a convenient form in terms of d. This is a preliminary report of our recent researches in this area.

Contents

§ 1. Introduction

An additive function f is an arithmetical function that satisfies the relation

$$f(m.n) = f(m) + f(n)$$

whenever the integers m and n are relatively prime. Thus, an additive function f is completely determined if one knows its values on all the integral powers of primes, p^{α}. By a special set of integers we simply mean any given subset of the positive integers.

The problem we discuss here is the distribution of values of additive functions $f(n)$ when $n \in S$. For the sake of simplicity we concentrate only on strongly additive functions which satisfy the additional condition $f(p^{\alpha}) = f(p)$, for all α, or equivalently

$$f(n) = \sum_{p|n} f(p). \tag{1.1}$$

The results we present can be suitably modified for general additive functions.

The question that first comes to mind is one concerning the average value of f for $n \in S$. With this in mind, and for more general considerations, we define

$$S_d(x) = \sum_{\substack{1 \leq n \leq x \\ n \in S,\ n \equiv 0 (\mathrm{mod}\ d)}} a_n \quad , \quad S(x) = S_1(x), \tag{1.2}$$

where the a_n are positive weights associated with $n \in S$. It is clear from (1.1) and (1.2) that

$$\sum_{\substack{1 \le n \le x \\ n \in S}} a_n f(n) = \sum_{p \le x} f(p)\, S_p(x) . \tag{1.3}$$

So, in order to estimate the expression in (1.3), or treat related sums, it is only natural to consider sets S for which $S_d(x)$ is given conveniently in terms of $S(x)$. More precisely we will discuss sets S and weights a_n such that

$$S_d(x) = \frac{S(x)\ \omega(d,x)}{d} + E_d(x), \tag{1.4}$$

where $\omega(d,x)$ satisfies some mild conditions, and the average of E_d is in some sense small.

Since the values of an additive function purely depend on its values on prime powers, a second class of sets that arise quite naturally are those S such that

$$S = \left\{ n \in \mathbb{Z} \;\middle|\; p^{\alpha} \| n \Rightarrow p^{\alpha} \in \wp \right\}, \tag{1.5}$$

where $\wp$ is a certain finite or infinite set of prime powers and $p^{\alpha} \| n$ means that $p^{\alpha} \mid n$ and $p^{\alpha+1} \nmid n$. For certain $\wp$ the S so generated satisfies a condition like (1.4) but in general it is even non-trivial to establish an analogue of (1.4).

Our goal is to estimate the average of f for $n \in S$ and obtain an upper bound for the deviation of f about its average.

In some cases we discuss the finer aspects of the distribution of f about its mean **value**. We do not claim to solve these questions for cases (1.4) and (1.5) in any complete form. This is only a preliminary report of our researches in the area, where some of our results are presented in the context of classical work. Owing to this reason, we discuss some special sets in greater detail. The proofs of only some of the results are discussed and even here we highlight only the main ideas. Details will appear elsewhere in separate papers.

There is a vast literature on the distribution of additive functions over the set of positive integers. The subject has its origins in the fundamental observation of Hardy and Ramanujan in 1917 [15], regarding the additive function $\nu(n)$, the number of prime factors of n. They showed that $\nu(n)$ has average value log log n and that for almost all n the deviation about log log n is less than $(\log\log n)^{1/2+\varepsilon}$ for every $\varepsilon > 0$. It is interesting to note now that this was one paper of Hardy-Ramanujan that did not have immediate impact! Actually it was only in 1934 that the probabilistic nature of their result was realised when Turán (see Hardy and Wright [16], p.356-58) gave a simple **proof** of it. In 1940, Erdös and Kac [9] proved more generally, that for an additive function satisfying some mild conditions, one could define a distribution function $F_x(\lambda)$ on the set $\{f(n) | 1 \leq n \leq x\}$ such that $F_x(\lambda)$ tends to the normal distribution as $x \longrightarrow \infty$. (Further light on the work of Erdös-Kac and Turán will be shed in § 2 and § 3). From then on a tremendous amount of work has been done on the distribution of additive functions for integers n satisfying $1 \leq n \leq x$ (see Elliott's book [7]).

The original results of Erdös-Kac and Turán have been extended to certain special sets such as

$$S = \{g(n) \mid n \in \mathbb{Z}^+\}, \; g(x) \in \mathbb{Z}[x] \tag{1.6}$$

and

$$S = \{p + 1 \mid p = \text{prime}\} . \tag{1.7}$$

These are examples of sets S that satisfy (1.4). To our knowledge only Barban [3] attempted a general discussion of sets satisfying conditions like (1.4). The results we present are however quite different from his.

As far as we know only Levin-Fainleib [18] have considered amongst other things sets of the type (1.5). Their methods though highly interesting are quite complicated; in addition there are some mathematical errors in [18] . Thus it seemed to us worthwhile to consider this question afresh.

In sections 2 and 3 we review some classical results and methods. Our new results are presented from section 4 onwards.

All the notation introduced so far will be retained. In addition we let

$$G(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\lambda} e^{-u^2/2} du. \qquad (1.6)$$

The notations 'O' and $\ll$ are used interchangeably as in convenient. Implicit constants are absolute unless otherwise indicated. By c_i, and $c_i(r)$, $i = 1,2,\ldots,$ we mean absolute positive constants and positive functions of r respectively. For a positive integer n we let $p(n)$ and $P(n)$ denote respectively the smallest and largest prime factors of n if $n > 1$, and put $p(1) = \infty$ and $P(1) = 1$. Also, $\pi(x) = \sum_{p \leqslant x} 1$ and

$$li(x) = \int_{2}^{\max(2,x)} \frac{dt}{\log t} .$$

The term 'almost all' is used in broader sense " all but an insignificant number of ", instead of in the narrower sense - " all but a finite number of" . Finally, a multiplicative function $h(n)$ is one that satisfies $h(m.n) = h(m).h(n)$ for $(m,n) = 1$.

§ 2. The Turán-Kubilius inequality and its dual

An arithmetical function f has g as its normal order if $g(n)$ is monotonic and

$$\lim_{x \longrightarrow \infty} \frac{N_\varepsilon(x)}{x} = 1, \qquad (2.1)$$

where

$$N_{\varepsilon}(x) = \sum_{\substack{1 \leq n \leq x \\ 1-\varepsilon < \frac{f(n)}{g(n)} < 1+\varepsilon}} 1 \quad . \tag{2.2}$$

Hardy and Ramanujan [15] established that $\nu(n)$ has normal order log log n by obtaining upper bounds for

$$\nu_k(x) = \sum_{\substack{1 \leq n \leq x, \\ \nu(n) = k}} 1 \quad . \tag{2.3}$$

They needed only to show that $\nu_k(x)$ is small when k is well removed from log log x. The precise result they established by induction on k was

$$\nu_k(x) << \frac{x\,(\log\log x)^{k-1}}{(k-1)!\,\log x} \quad , \quad k = 1,2,3,\ldots \quad . \tag{2.4}$$

They combined (2.4) with the observation that for all $\delta > 0$

$$\lim_{t \to \infty} e^{-t} \sum_{\substack{n=1 \\ |n-t| > t^{1/2+\delta}}}^{\infty} \frac{t^n}{n!} = 0,$$

to deduce that for almost all integers n, we have

$|\nu(n) - \log\log n| < (\log\log n)^{1/2+\delta}$.

Paul Turán proved several years later (see [16] , p.356-58) that

$$\sum_{1 \leq n \leq x} (\nu(n) - \log\log x)^2 << x \log\log x . \qquad (2.5)$$

Thus the number of integers $n \leq x$ for which $|\nu(n) - \log\log n| > \lambda\sqrt{\log\log n}$ is $<< x/\lambda^2$. So $\nu(n)$ has normal order $\log\log n$.

Turán's method puts the Hardy-Ramanujan results in a probabilistic setting since (2.5) is essentially an upper bound for the second moment of $\nu(n)$. It was soon realised that inequalities like (2.5) might hold more generally for other additive functions. Turán himself [23] generalised (2.5) to real valued additive functions satisfying $0 \leq f(p^\alpha) << 1$. However it was only in 1956 that a complete and proper generalisation of (2.5) was given by Kubilius [17] .

THEOREM 1. (Turán-Kubilius inequality) *Let f(n) be a strongly additive function and let*

$$A(x) = \sum_{p \leq x} \frac{f(p)}{p} , \quad B(x) = \sum_{p \leq x} \frac{|f(p)|^2}{p} .$$

Then

$$\sum_{n \leq x} |f(n) - A(x)|^2 << x\, B(x). \quad // \qquad (2.6)$$

The importance of Theorem 1 stems from the fact that it holds without any conditions on the values $f(p)$. This enables one to derive a dual of (2.6) as the following lemma from linear algebra shows.

<u>LEMMA 1.</u> <u>Let</u> $c_{i,j}$ <u>be</u> mn <u>complex numbers for</u> $1 \leq i \leq m$, $1 \leq j \leq n$. <u>Assume that for all</u> n-<u>tuples of complex</u> numbers $x_1, x_2, \ldots, x_n$ <u>we have</u>

$$\sum_{i=1}^{m} \left| \sum_{j=1}^{n} c_{i,j} \, x_j \right|^2 \leq \lambda \sum_{j=1}^{n} |x_j|^2 \tag{2.7}$$

<u>Then for all m-tuples of complex numbers</u> $y_1, \ldots, y_m$ <u>we have</u>

$$\sum_{j=1}^{n} \left| \sum_{i=1}^{m} c_{i,j} \, y_i \right|^2 \leq \lambda \sum_{i=1}^{m} |y_i|^2 \, . \, // \tag{2.8}$$

We now apply Lemma 1 to Theorem 1 with c_{ij} replaced by

$$c(p,n) = \begin{cases} \sqrt{p} - \frac{1}{\sqrt{p}} & \text{if } p \mid n \\ -\frac{1}{\sqrt{p}} & \text{if } p \nmid n \end{cases}$$

and x_j replaced by

$$x_p = f(p)$$

for $1 \leq p \leq x$, $1 \leq n \leq x$. The inequality (2.6) is exactly (2.7) with $f(p)$ replaced by $f(p)\sqrt{p}$ and an appropriate λ. We give below the suitable dual to (2.6) by appealing to (2.8).

<u>THEOREM 2.</u> <u>For any sequence</u> b_n <u>of complex numbers</u> <u>we have</u>

$$\sum_{p \leq x} p \left| \sum_{\substack{1 \leq n \leq x \\ n \equiv 0 \pmod p}} b_n - \frac{1}{p} \sum_{1 \leq n \leq x} b_n \right|^2 \ll x \sum_{1 \leq n \leq x} |b_n|^2 . // \tag{2.9}$$

(For a proof of Theorems 1 and 2 and Lemma 1, see Elliott [7], Ch. 4).