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CONTENTS

APPROXIMATION TO NONLINEAR CONVECTION DIFFUSION EQUATIONS

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We try to build a strong numerical method for convection diffusion equations in the nonlinear case, which gives L^∞ and Bounded Variation (=BV) stability on the gradient of the solution. This leads to a compactness argument in L^∞ for the approximate solution and then to a proof of convergence on a nonlinear diffusion term. Several examples are reported in order to show that hyperbolic techniques are suitable for such nonlinear parabolic models.

This paper is divided into 3 parts. A first example is detailed in Section 1, where the numerical method is described in a very simple way. Then the same method is adapted to the porous media equation in Section 2. Next, Section 3 is devoted to the two dimension case, including some numerical techniques adapted to the diffusion term. Then a Riemann solver is proposed for the first order term, which comes from the derivation of the diffusion term. This leads to a two dimension version of the Lax Friedrichs scheme, and a construction of the Godunov scheme using the same Riemann solver.

Other numerical methods presents same properties of stability, such as the one proposed in [6],[8],[9],[10] or [12]. However, the mathematical model studied here deals with the equation of velocity and the schemes proposed here too.

1.- AN EXAMPLE - We consider the equation

$$u_t = \left(u\, u_x \right)_x \tag{1}$$

together with the initial condition

$$u(x,0) = u_0(x)$$

where

$$u_0 \in W^{1,\infty}(\mathbb{R}) \ , \ u_0 \geq 0 \ , \quad \text{with compact support.}$$

For $i \in \mathbb{Z}$ and $n \in \mathbb{N}$, we denote by u_i^n the approximate value of $u(i\Delta x, n\Delta t)$, for a space increment Δx and a time increment Δt.
By the Euler scheme, we get

$$u_i^{n+1} = u_i^n + \frac{\Delta t}{2\Delta x^2} \left[(u_{i+1}^n + u_i^n)(u_{i+1}^{n+1} - u_i^{n+1}) - (u_i^n + u_{i-1}^n)(u_i^{n+1} - u_{i-1}^{n+1})\right] \qquad (2)$$

which leads to the estimates, provided all u_i^n to be non negative,

$$\forall\, i \in \mathbb{Z} \quad , \quad u_i^{n+1} \geq 0 \quad ,$$

$$\operatorname{Max}_{i \in \mathbb{Z}} \left| u_i^{n+1} \right| \leq \operatorname{Max}_{j \in \mathbb{Z}} \left| u_j^n \right| ,$$

and

$$\sum_{i \in \mathbb{Z}} \left| u_{i+1}^{n+1} - u_i^{n+1} \right| \leq \sum_{j \in \mathbb{Z}} \left| u_{j+1}^n - u_j^n \right| \quad .$$

This means that the scheme repserves the positiveness of u , and is L^∞ and BV (=Bounded Variation) stable (or is TVD, that is Total Variation Diminishing).

Let ψ be a test function in $C^2(\mathbb{R} \times \mathbb{R}_+)$. We set

$$\psi_i^n = \psi(i\Delta x, n\Delta t)$$

and

$$u_{i+1/2}^n = \frac{1}{2} \left(u_i^n + u_{i+1}^n \right) \quad .$$

Then we get, by multiplying the scheme (2) by ϕ_i^n and summing,

$$\sum_{i,n} u_i^n \frac{\psi_i^n - \psi_i^{n-1}}{\Delta t} \Delta t \Delta x = \sum_{i,n} u_{i+1/2}^n \frac{u_{i+1}^{n+1} - u_i^{n+1}}{\Delta x} \frac{\psi_{i+1}^n - \psi_i^n}{\Delta x} \Delta t \Delta x$$

However the estimates given above are not sufficient to enable us to go to the limit (for any subsequence) on the product

$$u_{i+1/2}^n \frac{u_{i+1}^{n+1} - u_i^{n+1}}{\Delta x} \quad .$$

We need another estimate, which can be the uniform convergence on u . This can be deduced from L^∞ and BV estimates on u_x instead of u as above.

In order to get it, we set

$$v = - u_x \ .$$

Then we get the equation

$$v_t + (v^2)_x = (u\, v_x)_x \ . \qquad (3)$$

This equation will be discretized into two steps. The first one is devoted to the second order term, which correspond to a diffusion term. We set $m = n + 1/2$, which will be used as the upper index for an intermadiary value between the times $n\Delta t$ and $(n+1)\Delta t$. We compute

$$v_i^m = v_i^n + \frac{\Delta t}{\Delta x^2} \left[u_{i+1/2}^n (v_{i+1}^m - v_i^m) - u_{i-1/2}^n (v_i^m - v_{i-1}^m) \right] \tag{4}$$

As above, this scheme preserves L^∞ and BV estimates for v . It is now sufficient to use a L^∞ and BV stable scheme for the discretization of the first order term. This can be done by using the Godunov scheme.

This scheme uses a Riemann solver associated with the scalar equation

$$v_t + (v^2)_x = 0 \tag{5}$$

which allows to compute the fluxes on both sides of the cells.
This is performed as follows. We compute, for any $i \in \mathbb{Z}$,

$$v_{i+1/2}^m = \begin{cases} v_i^m & \text{if } v_i^m \geq 0 \text{ and } v_i^m + v_{i+1}^m \geq 0 , \\ 0 & \text{if } v_i^m \leq 0 \text{ and } v_{i+1}^m \geq 0 , \\ v_{i+1}^m & \text{if } v_{i+1}^m \leq 0 \text{ and } v_i^m + v_{i+1}^m \leq 0 . \end{cases}$$

and then,

$$v_i^{n+1} = v_i^m - \frac{\Delta t}{\Delta x} \left[(v_{i+1/2}^m)^2 - (v_{i-1/2}^m)^2 \right] \tag{6}$$

This scheme is $L^\infty$and BV stable under the stability condition

$$\operatorname{Max}_j |v_j^m| \frac{\Delta t}{\Delta x} \leq \frac{1}{2} \tag{7}$$

We notice that this condition is the well known Courant Friedrichs Lewy condition, and the coefficient $\frac{1}{2}$ comes from the flux in (5), which is $2v$. This condition gives also the conservation of the positiveness of u , since we have

$$u_{i+1/2}^{n+1} = -\sum_{j \leq i} v_j^{n+1} \Delta x \geq 0 .$$

As a matter of fact,

$$-\sum_{j \leq i} v_j^{n+1} = -\sum_{j \leq i} v_j^m + \frac{\Delta t}{\Delta x} (v_{i+1/2}^m)^2 \geq - w_{i+1/2}^m$$

by writing

$$w_{i+1/2}^m = \sum_{j \leq i} v_j^m .$$

And since we have

$$w^m_{i+1/2} = w^n_{i+1/2} + \frac{\Delta t}{\Delta x} u^n_{i+1/2}\left(w^m_{i+3/2} - 2\, w^m_{i-1/2} + w^m_{i-1/2}\right)$$

which is a linear system involving a M-matrix, we get

$$\forall\ i \in \mathbb{Z} \quad w^n_{i+1/2} \le 0 \quad \Rightarrow \quad \forall\ i \in \mathbb{Z} \quad w^m_{i+1/2} \le 0\ .$$

This proves the conservation of the positiveness.

2.- THE POROUS MEDIA EQUATION. We are now concerned with the equation

$$\frac{\partial u}{\partial t} = \Delta\ \Phi(u) \tag{8}$$

where $\Phi \in C^2(\mathbb{R})$ is a nondecreasing function such that $\Phi'(0)=0$. We set

$$\phi(u) = \Phi'(u)\ ,$$

and

$$v = - \frac{\phi(u)}{u} \frac{\partial u}{\partial x}\ .$$

Then we get the convection equation

$$u_t + (\ u\ v\ )_x = 0\ . \tag{9}$$

Next we introduce

$$p(u) = \int_0^u \frac{\phi(y)}{y}\, dy$$

which corresponds physically to a pressure if u is a concentration. Then we get

$$v + p_x = 0 \tag{10}$$

which is known as the Darcy law; here v is a velocity.

From (9) and by using (10), we can derive the equation of the velocity and obtain

$$v_t + (v^2)_x = (\ \phi(u)\ v_x\ )_x \quad . \tag{11}$$

We propose a discretization of this equation.

Since ϕ is nonnegative, the previous scheme will work and we get L^∞ and BV estimates for v . We notice that in the first step (i.e. the discretization of the second order term), we only have to change $u^n_{i+1/2}$ into $\phi(u^m_{i+1/2})$, which can be written as a function of the pressure p by using the Darcy law (10). This will be denoted $d(p) = \phi(u)$, and for example

$$d(p) = \frac{p}{k} \qquad \text{if} \qquad \phi(u) = u^k .$$

This is possible only when the positiveness of the pressure is preserved during the two steps of the scheme. By writing, from the Darcy law (10),

$$p^m_{i+1/2} = p^m_{i-1/2} - v^m_i \Delta x \tag{12}$$

and

$$\phi^n_{i+1/2} = d(p^n_{i+1/2}) ,$$

we get

$$p^m_{i+1/2} = p^n_{i+1/2} + \frac{\Delta t}{\Delta x} \phi^n_{i+1/2} \left[p^m_{i+3/2} - 2p^m_{i+1/2} + p^m_{i-1/2} \right]$$

which involves a M-matrix. Then we get

$$\forall\ j \in \mathbb{Z} \quad p^n_{j+1/2} \geq 0 \quad \Rightarrow \quad \forall\ i \in \mathbb{Z} \quad p^m_{i+1/2} \geq 0 .$$

Next we have, by computing $p^{n+1}_{i+1/2}$ from the v^{n+1}_j as for the intermediary values in (12),

$$p^{n+1}_{i+1/2} = p^m_{i+1/2} + \Delta t \ (v^m_{i+1/2})^2 \geq p^m_{i+1/2} \geq 0 .$$

From these estimates we can deduce the convergence of a subsequence, from a compactness argument, towards a weak solution which can be defined, for example, as follows.

For any test function ψ with a compact support in $\mathbb{R}\times\mathbb{R}_+$, v and p satisfy

$$\iint_{\mathbb{R}\times\mathbb{R}_+} (v\, \psi_t + v^2 \psi_x)\ dx\ dt = \iint_{\mathbb{R}\times\mathbb{R}_+} (d'(p)\ v^2\ \psi_x - d(p)\ v\ \psi_{xx})\ dx\ dt$$

and

$$\iint_{\mathbb{R}\times\mathbb{R}_+} (v\ \psi - p\ \psi_x)\ dx\ dt = 0 .$$

Here the convergence on each product is possible since we have a uniform convergence for p and a strong L^1 convergence for v.

3. - THE TWO DIMENSION CASE. For a given non negative function ϕ in $C^1(\mathbb{R})$, with $\phi(0) = 0$, we consider the two dimension equation

$$w_t = \mathrm{div}(\phi(w) \nabla w) \tag{13}$$

The two space variables will be denoted x and y. We set

$$V = \begin{pmatrix} u \\ v \end{pmatrix} = - \frac{\phi(w)}{w} \nabla w.$$

Then for

$$p = \int_0^w \frac{\phi(\xi)}{\xi} d\xi \tag{14}$$

we obtain the Darcy law

$$V + \nabla p = 0, \tag{15}$$

and the convection equation

$$w_t + \mathrm{div} (w V) = 0. \tag{16}$$

Here p corresponds to a pressure, V is a velocity field and w is a concentration. This is physically meaningful when p and w has nonnegative values.

As above in the one dimension case, we compute the time derivative of V . We obtain successively,

$$\begin{aligned} V_t &= - \nabla p_t && \text{from (15)}, \\ &= - \nabla \left\{ \frac{\phi(w)}{w} w_t \right\} && \text{by using (14)}, \\ &= + \nabla \left\{ \frac{\phi(w)}{w} \mathrm{div}(w V) \right\} && \text{by using (16)}. \end{aligned}$$

From

$$\mathrm{div} (w V) = V . \nabla w + w \,\mathrm{div} (V) ,$$

we get the equation of the velovity

$$V_t + \nabla (|V|^2) = \nabla (d(p) \,\mathrm{div}(V)) \tag{17}$$

since from (14) we can find a function d of the pressure such that

$$d(p) = \phi(w) .$$

We propose now a two step numerical scheme for the discretization of (17). The first step deals with the second order term. We can use here a classical technique for diffusion equations.

For example, we can write

$$q = \mathrm{div}(V)$$

which satisfies

$$q_t = \Delta \; (\; d(p) \; q \;) \; .$$

This equation can be solved by using the implicit Euler method for the time discretization, and a finite difference method with a frosen d(p) for the space discretization. Then it remains to solve

$$\mathrm{div}\;(V) = q \qquad ; \qquad \mathrm{rot}(V) = 0 \quad ,$$

to get the velocity. Such an elliptic problem is studied in [3] or [5].

Other classical techniques can work too. Now, from this first step, we get intermediary values $V^m_{i,j}$ of the velocity field on each cell

$$M_{i,j} = \left] \; (i-\tfrac{1}{2})\Delta x \; , \; (i+\tfrac{1}{2})\Delta x \; \right[\times \left] (j-\tfrac{1}{2}) \; \Delta y \; , \; (j+\tfrac{1}{2})\Delta y \right[$$

where Δx and Δy denote the space meshsizes.

We are now concerned with the second step of the numerical scheme. This corresponds to a discretization of the non linear hyperbolic system

$$\begin{aligned} u_t + (\; u^2 + v^2 \;)_x &= 0 \quad , \\ v_t + (\; u^2 + v^2 \;)_y &= 0 \quad . \end{aligned} \tag{18}$$

Either for a Godunov scheme using alternated directions or for a Lax Friedrichs scheme with modified fluxes, we need a one dimension Riemann solver. In the x-direction for example, we have to solve the Riemann problem

$$\begin{aligned} u_t + \;\; (\; u^2 + v^2 \;) &= \;\; 0 \\ v_t \qquad &= \;\; 0 \quad , \end{aligned} \tag{19}$$

with the constant piecewise intial condition

$$(u(x,0), v(x,0)) = \begin{cases} (u_l, v_l) & \text{for} \quad x < 0 \; , \\ (u_r, v_r) & \text{for} \quad x > 0 \; , \end{cases}$$

where u_l, v_l, u_r, v_r are given real data. We can have either a wave travelling towards the right hand side (with a positive velocity), or a wave travelling towards the left hand side (with a negative velocity). This wave can be either a rarefaction wave or a shock wave.

In both cases we have a constant state (u,v_r) (in the first case) or (u,v_l) (in the second case) between the line x=0 and the wave.

This value u satisfies the following condition, which is a Rankine Hugoniot condition along the line x=0,

$$u^2 + v_r^2 = u_l^2 + v_l^2 \qquad \text{(first case)}$$

$$u^2 + v_l^2 = u_r^2 + v_r^2 \qquad \text{(second case)} \tag{20}$$

For a shock wave with a positive velocity, we have necessary, from an entropy argument,

$$u + u_r > 0 \qquad \text{and} \qquad u > 0 \; .$$

Then we need in this case,

$$u_l^2 + v_l^2 \geq u_r^2 + v_r^2 \qquad \text{with} \qquad u \geq 0 \; .$$

For a rarefaction wave with a positive velocity, we have

$$0 \leq u \leq u_r$$

then we need

$$u_l^2 + v_l^2 \leq u_r^2 + v_r^2 \qquad \text{with} \qquad u \geq 0 \; .$$

For a shock wave with a negative velocity we have

$$u_l^2 + v_l^2 \leq u_r^2 + v_r^2 \qquad \text{with} \qquad u \leq 0 \; .$$

For a rarefaction wave with a negative velocity we have

$$u_l^2 + v_l^2 \geq u_r^2 + v_r^2 \qquad \text{with} \qquad u \geq 0 \; .$$

Another case can arise, when a rarefaction wave is spread on both sides of the line x=0 . This is very seldom in practice, and we have

$$u_l < 0 < u_r \qquad \text{and} \qquad |v_l| = |v_r| \; .$$

From these remarks, we can solve the Riemann problem as follows.

If $u_l^2 + v_l^2 \geq u_r^2 + v_r^2$ then we have

if $v_l^2 \leq u_r^2 + v_r^2$ and $u_l < 0$, a rarefaction wave with a negative velocity,

else a shock wave with a positive velocity.

If $u_l^2 + v_l^2 \leq u_r^2 + v_r^2$ then we have

if $v_r^2 \leq u_l^2 + v_l^2$ and $u_r > 0$, a rarefaction wave with a positive velocity,

else a shock wave with a negative velocity.

Using this Riemann solver we are now able to compute

$$F_x(u_l, v_l, u_r, v_r) = \begin{cases} u_l^2 + v_l^2 & \text{(in the first case)} \\ u_r^2 + v_r^2 & \text{(in the second case)} \end{cases}$$

as for the first or the second case in (20), and

$$G(u,v,u,v) = \frac{2}{\Delta t \ \Delta x} \int_{-\Delta x/2}^{\Delta x/2} \int_0^{\Delta t/2} \left[u(x,t)^2 + v(x,t)^2 \right] dx \, dt \ .$$

By the same way, a Riemann solver associated with the Riemann problem

$$\begin{aligned} u_t &= 0 \ , \\ v_t + (u^2+v^2)_y &= 0 \ , \end{aligned} \qquad (21)$$

$$(u(x,0), v(x,0)) = \begin{cases} (u_l, v_l) & \text{for } y < 0 \ , \\ (u_r, v_r) & \text{for } y > 0 \ . \end{cases}$$

can be built and we are also able to compute

$$F_y(u_l, v_l, u_r, v_r) \quad \text{and} \quad G_y(u_l, v_l, u_r, v_r) \ .$$

We can notice that the same Riemann solver can be used since we have only to change u into v and v into u .

Now we can write the Godunov scheme as follows,

$$F^m_{i+1/2,j} = F_x(u^m_{i+1,j}, v^m_{i+1,j}, u^m_{i,j}, v^m_{i,j}) ,$$

$$F^m_{i,j+1/2} = F_y(u^m_{i,j+1}, v^m_{i,j+1}, u^m_{i,j}, v^m_{i,j}) ,$$

$$u^{n+1}_{i,j} = u^m_{i,j} - \frac{\Delta t}{\Delta x} (F^m_{i+1/2,j} - F^m_{i-1/2,j}) ,$$

$$v^{n+1}_{i,j} = v^m_{i,j} - \frac{\Delta t}{\Delta y} (F^m_{i,j+1/2} - F^m_{i,j-1/2}) ,$$

or the modified Lax Friedrichs scheme as follows

$$G^m_{i+1/2,j} = G_x(u^m_{i+1,j}, v^m_{i+1,j}, u^m_{i,j}, v^m_{i,j})$$

$$G^m_{i,j+1/2} = G_y(u^m_{i,j+1}, v^m_{i,j+1}, u^m_{i,j}, v^m_{i,j})$$

$$u^\mu_{i+1/2,j+1/2} = \frac{1}{4} (u^m_{i,j} + u^m_{i,j+1} + u^m_{i+1,j} + u^m_{i+1,j+1}) - \frac{\Delta t}{\Delta x} (G^m_{i+1,j+1/2} - G^m_{i,j+1/2})$$

$$v^\mu_{i+1/2,j+1/2} = \frac{1}{4} (v^m_{i,j} + v^m_{i+1,j} + v^m_{i,j+1} + v^m_{i+1,j+1}) - \frac{\Delta t}{\Delta y} (G^m_{i+1/2,j+1} - G^m_{i+1/2,j})$$

which corresponds to new intermediary values denoted by μ), and

$$G^\mu_{i,j+1/2} = G_x(u^\mu_{i+1/2,j+1/2}, v^\mu_{i+1/2,j+1/2}, u^\mu_{i-1/2,j+1/2}, v^\mu_{i-1/2,j+1/2})$$

$$G^\mu_{i+1/2,j} = G_y(u^\mu_{i+1/2,j+1/2}, v^\mu_{i+1/2,j+1/2}, u^\mu_{i+1/2,j-1/2}, v^\mu_{i+1/2,j-1/2})$$

$$u^{n+1}_{i,j} = \frac{1}{4}(u^\mu_{i+1/2,j+1/2} + u^\mu_{i+1/2,j-1/2} + u^\mu_{i-1/2,j+1/2} + u^\mu_{i-1/2,j-1/2}) - \frac{\Delta t}{\Delta x} (G^\mu_{i+1/2,j} - G^\mu_{i-1/2,j})$$

$$v^{n+1}_{i,j} = \frac{1}{4} (v^\mu_{i+1/2,j+1/2} + v^\mu_{i+1/2,j-1/2} + v^\mu_{i-1/2,j+1/2} + v^\mu_{i-1/2,j-1/2}) - \frac{\Delta t}{\Delta y} (G^\mu_{i,j+1/2} - G^\mu_{i,j-1/2})$$

This scheme has been studied for the scalar equation in two dimensions and a proof of convergence is given in [2]. Here the fluxes are different from those given in [4], by Conway and Smoller. In this case the authors use an average in the two directions, which spreads out the approximate solution near a singularity.

-CONCLUSION- The idea which consists in taking out a convection term from the term of nonlinear diffusion is not really a new one (see e.g. [7]). We think that to apply this method to the equation of the velocity seems to be a new one. This can be also adapted to the case of a second arder term which is not a degenerated one or when a convection term already lies in the equation . This is the case in interdiffusion problems(see [1]), in cellular division, population dynamics and many other topics. This technique has a good behaviour near a the free boundary corresponding to the degeneration point (here for u = 0), or when a boundary condition is to be taken in account. A Riemann solver of the same conception appears in some problems of combustion and allows to introduce pointwise boundary conditions (see [11]). An antidiffusion technique adapted to this version of the Lax Friedrichs method has been analysed in [2] ; sufficient conditions for convergence are given for the quasi linear equation.

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DIFFERENCE SCHEMES FOR NONLINEAR HYPERBOLIC SYSTEMS - A GENERAL FRAMEWORK

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ABSTRACT

For a hyperbolic system of conservation laws, the general form of conservative difference schemes involving two time-levels in an explicit or implicit way is obtained under natural assumptions. General results are shown on the schemes and this framework is used to study implicit schemes of second-order accuracy.

1. INTRODUCTION

After the pioneering works of Lax and Godunov in the late fifties, a lot of conservative difference schemes have been proposed for the solution of hyperbolic systems of conservation laws, using either a centred approximation in space or some upwinding. Since the late seventies, the schemes devised have been mostly implicit and free of severe stability constraints on the time step. Nowadays, a great number of explicit and implicit schemes are available and it would be useful to gather them in a general framework in order to unify their presentation, simplify their analysis, and make the search easier for new efficient methods. The aim of the present paper is to propose such a framework in the case of conservative schemes involving only two time - levels.

The construction of the general form of the schemes is developed in Section 2. Then some examples are given in Section 3 showing that the usual schemes can be simply identified. In Section 4, necessary and sufficient conditions are presented to obtain second-order accuracy, solvability, stability, dissipation and diagonal dominance. Section 5 describes the application of the general framework to the study of implicit schemes of second - order accuracy. Possible developments are indicated in the conclusions.

2. CONSTRUCTION OF THE TWO-LEVEL SCHEMES

Let us consider an initial-value problem for the system of m conservation laws :

$$w_t + f(w)_x = 0, \qquad x\in\mathbb{R}, \quad t > 0, \tag{2.1}$$

where the state-vector $w(x,t)$ belongs to an open set Ω of $\mathbb{R}^m$ and the flux - function $f : \Omega \to \mathbb{R}^m$ is smooth. This system can also be written in the expanded form :

$$w_t + A(w) \; w_x = 0 \tag{2.2}$$

with the jacobian matrix $A(w) = df(w) / dw$.

System (2.1) is assumed to be hyperbolic, i.e. the matrix $A(w)$ has m real eigenvalues and a complete set of eigenvectors.

We approximate System (2.1) by a finite-difference scheme with 2 time - levels :

$$S \; (w_{j-J}, w_{j-J+1}, \ldots, w_{j+J} \; ; \; \Delta w_{j-J_1}, \ldots, \Delta w_{j+J_1} \; ; \; \sigma) = 0 \tag{2.3}$$

where $w_j \equiv w_j^n$ is the numerical solution at the old time-level $t = n \, \Delta t$ for $x = j\Delta x$, $\Delta w_j \equiv w_j^{n+1} - w_j^n$ is the increment of the numerical solution during a time step and σ denotes the step ratio :

$$\sigma = \Delta t \; / \; \Delta x.$$

Scheme (2.3) involves (at most) $2J+1$ points at the old time-level and $2J_1+1$ points at the new one. It is explicit if $J_1 = 0$ or implicit otherwise.

Scheme (2.3) is assumed to be **conservative**, which means it can be written as :

$$\Delta w_j = - \sigma \; (h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}}) \tag{2.4}$$

with a numerical flux :

$$h_{j+\frac{1}{2}} = h \; (w_{j-J+1}, \ldots, w_{j+J} \; ; \; \Delta w_{j-J_1+1}, \ldots, \Delta w_{j+J_1} \; ; \; \sigma)$$

where h is a Lipschitz-continuous function satisfying the consistency condition :

$$h \; (u, u, \ldots, u \; ; \; 0, 0, \ldots, 0 \; ; \; \sigma) = f \; (u), \qquad u\in\Omega. \tag{2.5}$$

Similarly as in the Lax and Wendroff paper for explicit schemes [1], one can easily show that if such a conservative scheme converges boundedly almost everywhere as Δx and Δt tend to zero, then it converges to a weak solution of System (2.1).

Furthermore, we assume that the scheme (2.4) involves **essentially 3 points**, i.e.

$$h\,(u_{-J+1}, \ldots, u_{-1}, u, u, u_2, \ldots, u_J ;$$
$$v_{-J_1+1}, \ldots, v_{-1}, 0, 0, v_2, \ldots, v_{J_1} ; \sigma) = f\,(u)\ , \qquad (2.6)$$

for any $u \in \Omega$, $u_p \in \Omega$ and $v_p \in \mathbb{R}^m$.

This property, first introduced for explicit schemes (see [2, Section 4] and [3]), is stronger than the consistency condition (2.5) , but it is satisfied by nearly all the schemes presently used in practice. Roughly speaking, it means that the consistency is ensured with the 3 central-points.

Finally, the scheme (2.4) is supposed to be either **explicit or linearly implicit**, i.e. its numerical flux is of the form :

$$h_{j+\frac{1}{2}} = h^{\exp l}_{j+\frac{1}{2}} + \sum_{p=-J_1+1}^{J_1} (H_p)_{j+\frac{1}{2}}\ \Delta w_{j+p} \qquad (2.7)$$

with an explicit part :

$$h^{\exp l}_{j+\frac{1}{2}} = h^{\exp l}\,(w_{j-J+1}, \ldots, w_{j+J}; \sigma)$$

and an implicit part with mxm matrix coefficients :

$$(H_p)_{j+\frac{1}{2}} = H_p\,(w_{j-J+1},\ldots,w_{j+J}; \sigma), \qquad p=-J_1+1, \ldots, J_1.$$

With the numerical flux (2.7), the scheme (2.4) leads to the solution of an algebraic linear system at each time iteration. For simplicity reasons, all the usual schemes are explicit or linearly implicit.

Let us now give the general form of the schemes satisfying the above assumptions. To write this form down, we need two classical operators for the space-differencing :

$$(\mu\psi)_{j+\frac{1}{2}} \equiv \frac{1}{2}(\psi_j + \psi_{j+1})$$

$$(\delta\psi)_{j+\frac{1}{2}} \equiv \psi_{j+1} - \psi_j$$

where ψ_j is a mesh function defined at $x = j\Delta x$ for integer values of $2j$.

Theorem 1 - Any conservative scheme (2.4) involving essentially 3 points and being explicit or linearly implicit can be written in the simple form :

$$\Delta w_j + \frac{\sigma}{2}\,\delta\,[M\,\mu\,(\Delta w)]_j - \frac{1}{4}\,\delta\,[P\,\delta(\Delta w)]_j + \sigma\,(\delta h')_j = -\,\sigma\,\delta(\mu f)_j + \frac{1}{2}\,\delta\,(Q\,\delta w)_j \qquad (2.8)$$

with three mxm - matrices depending on the old time-level :

$$M_{j+\frac{1}{2}} = M\,(w_{j-J+1},\ \ldots,\ w_{j+J}\ ;\ \sigma)$$

$$P_{j+\frac{1}{2}} = P\,(w_{j-J+1},\ \ldots,\ w_{j+J}\ ;\ \sigma)$$

$$Q_{j+\frac{1}{2}} = Q\,(w_{j-J+1},\ \ldots,\ w_{j+J}\ ;\ \sigma)$$

and a m-vector :

$$h'_{j+\frac{1}{2}} = \sum_{p=-J_1+1}^{J_1} (\mathcal{H}_p)_{j+\frac{1}{2}}\,(\delta w_{j+\frac{1}{2}},\ \Delta w_{j+p})$$

where the $(\mathcal{H}_p)_{j+\frac{1}{2}}$ are bilinear applications depending on the old-time level :

$$(\mathcal{H}_p)_{j+\frac{1}{2}} = \mathcal{H}_p\,(w_{j-J+1},\ \ldots,\ w_{j+J}\ ;\ \sigma), \quad \text{for } p\neq 0 \text{ and } 1$$

$$(\mathcal{H}_o)_{j+\frac{1}{2}} = (\mathcal{H}_1)_{j+\frac{1}{2}} = 0.$$

Remarks :

a) If $J_1 = 1$ (only 3 points at the new time-level), then the most complicated term disappears :

$$h'_{j+\frac{1}{2}} = 0.$$

Such a scheme is entirely characterized by the data of M, P and Q.