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FOREWORD

The present volume is based on the lectures of invited speakers at the Fifth National School in Algebra, held at the Black Sea coast, near the city of Varna, from September 24 to October 4, 1986.

The preceding National Algebraic Schools in Bulgaria were held biannually starting in 1975, with the primary aim of introducing young algebraists to some specific areas. Accordingly, small numbers of lecturers were invited to give comprehensive accounts of their particular fields.

At the Fifth National School, the number of invited speakers was increased considerably, as was the range of topics covered. The lecturers were requested to give broad surveys, at an advanced level, on topics of current research. We are glad to express the gratitude of the participants and the organizers to the speakers for sharing their insight and enthusiasm for many beautiful algebraic problems.

A second objective of this School was to host an international conference in Algebra, which was organized in five Special Sessions, held in the afternoons. Thanks are due to all participants who gave short communications on their results.

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and normal two-dimensional CM-rings, as well as for rings of minimal multiplicity, see [HK] and [BHU]. We show in the STOP PRESS at the end of the paper that e.g. all hypersurface rings of characteristic zero admit Ulrich-modules.

The study of MCM-modules over a hypersurface ring leads to matrix factorizations as introduced by Eisenbud in [E]. Given a homogeneous polynomial $f \in S$, an equation $fE = \alpha\beta$, where α and β are square matrices with homogeneous polynomials as entries, is called a *matrix factorization* of f. (Here E denotes the unit matrix; for simplicity we write f for fE in the sequel.)

As an example of a matrix factorization, consider

$$X_1^2 + X_2^2 = \begin{pmatrix} X_1 & X_2 \\ X_2 & -X_1 \end{pmatrix} \begin{pmatrix} X_1 & X_2 \\ X_2 & -X_1 \end{pmatrix} .$$

According to Eisenbud's theory the MCM-modules over $R = S/(f)$ correspond to matrix factorizations $f = \alpha\beta$, and the linear MCM-modules to such factorizations for which α is a matrix of linear forms. The precise statement is given in section 4.

For a non-singular quadratic form f it is shown in the paper of Buchweitz-Eisenbud-Herzog [BEH], that there exists essentially just one matrix factorization $f = \alpha\beta$, where α and β are matrices of linear forms, and consequently there exist up to isomorphisms at most two indecomposable linear MCM-modules (one corresponding to α, the other to β). This result is obtained by considering the Clifford algebra C of f, and showing that there is as well a correspondence between the matrix factorizations of f and the $\mathbf{Z}/2\mathbf{Z}$-graded modules over C.

In this paper we imitate this method in order to obtain similar but somewhat weaker results for homogeneous polynomials $f \neq 0$ of degree $d > 2$.

In section 1 we introduce the notion of a *generalized Clifford algebra for* f. Let

$$V = \bigoplus_{i=1}^{n} k e_i$$

be an n-dimensional k-vectorspace with basis $e_1, \ldots, e_n$, then f defines a map $f : V \longrightarrow k$, by $f(x_1 e_1 + \ldots + x_n e_n) := f(x_1, \ldots, x_n)$ for all $x_i \in k$. A $\mathbf{Z}/d\mathbf{Z}$-graded algebra C together with a monomorphism $V \rightarrow C_1$ is called a generalized Clifford algebra, if C is generated by V, and $f(x) = x^d$ for all $x \in V$. It is not at all clear that such an object exists for f. Of course, the most natural thing to do is to form the tensor algebra $T(V)$ of V and to divide by the relations $x \otimes \cdots \otimes x - f(x)$. We call

$$C(f) := T(V)/\langle\{x \otimes \cdots \otimes x - f(x) | x \in V\}\rangle$$

the *universal Clifford algebra for* f.

This algebra was first introduced by N. Roby ([R]) in this generality. The special case of binary cubic forms was already studied by N. Heerema ([H]) in 1954.

Using Gröbner basis arguments we show in theorem 1.8 that the natural map $V \rightarrow C_1(f)$ is an inclusion.

Of course any other generalized Clifford algebra for f is a quotient of $C(f)$ by a homogeneous two-sided ideal a in $C(f)$ with $a \cap V = 0$.

The essential observation in this paper is that the $\mathbf{Z}/d\mathbf{Z}$-graded modules over a generalized Clifford algebra correspond to *linear* matrix factorizations

$$f = \alpha_0 \cdot \ldots \cdot \alpha_{d-1} \ ,$$

where $\alpha_0, \ldots, \alpha_{d-1}$ are square matrices of linear forms, and moreover that if $f = \alpha_0 \cdot \ldots \cdot \alpha_{d-1}$ corresponds to $M = \bigoplus_{i=0}^{d-1} M_i$, then the size of α_i equals $\dim_k M_i$, which is independent of i (theorem 1.3).

A similar observation was made by Roby ([R]). Not taking into account the $\mathbf{Z}/d\mathbf{Z}$-grading of $C(f)$ he assigns to each $C(f)$-module a factorization $f = \alpha^d$ of f as a pure power of a linear square matrix. Such factorizations have been studied thoroughly by L. N. Childs ([C]).

Unfortunately $\dim_k C(f)$ is infinite if and only if $n > 1$ and $d > 2$, as is shown in theorem 1.8. (In the case that Char $k > d$ this theorem has been shown by Childs ([C, theorem 3]), using the early results by Heerema ([H]).) As a consequence, it is not clear whether there exists a finite dimensional $\mathbf{Z}/d\mathbf{Z}$-graded module over $C(f)$ for an arbitrary form f of degree at least three in more than two variables. Thus we don't know whether all homogeneous forms have linear matrix factorizations in the above sense with finite matrices.

On the other hand, if we weaken the conditions on the factorization slightly, we easily obtain factorizations of f with finite matrices:

— If we do *not* demand the factorization to be *linear*, we may just choose a non-trivial MCM-module over R, for instance the $(n-1)$ th syzygy module Ω_R^{n-1} of k. To this module corresponds according to Eisenbud's theory a matrix factorization $f = \alpha\beta$. (Both matrices cannot have linear forms as entries, unless f is a quadratic form.)

— If we allow *non-square* matrices, we may by quite elementary means decompose the 1×1-matrix (f) into factors, all of whose entries are linear:
For $i = 1, \ldots, d-1$, let B_i be a $1 \times \binom{n+i-1}{i}$ row matrix, whose entries are all the different monomials of degree i in the variables $X_1, \ldots, X_n$. For $i \in \{1, \ldots, d-2\}$ there is a matrix α'_i, whose non-zero entries all are variables, such that

$$B_i \alpha'_i = B_{i+1} \ .$$

Finally, there is an $\binom{n+d-2}{d-1} \times 1$ column matrix β, such that$(f) = B_{d-1}\beta$. If we let $\alpha_0 = B_1$, $\alpha_i = \alpha'_i$ for $i = 1, \ldots, d-2$, and $\alpha_{d-1} = \beta$, then indeed

$$(f) = \alpha_0 \cdot \ldots \cdot \alpha_{d-1} \ .$$

Even though such factorizations may be useful in other contexts, they do not provide us with linear MCM-modules over the hypersurface ring $S/(f)$.

In section 2 we extend a result of Atiyah-Bott-Shapiro [ABS] to generalized Clifford algebras. It essentially says that the category of $\mathbf{Z}/d\mathbf{Z}$-graded modules over a generalized Clifford algebra C is equivalent to the category of C_0-modules. This result considerably simplifies the further considerations, and applied to matrix factorizations it gives a deeper insight into the relations among the factors of a factorization $fE = \alpha_0 \cdot \ldots \cdot \alpha_{d-1}$. For instance we are able to say which of the factors α_i that are equivalent to each other (corollary 2.7), or under which circumstances fE is a power of a single matrix (corollary 2.4).

Section 3 is devoted to the study of diagonal forms. Most of its results may be found (explicitly or implicitly) in [L] or in [C]; we however give an essentially self-contained presentation of the results and proofs.

For the diagonal forms, finite-dimensional generalized Clifford algebras may be constructed. Just as for quadratic forms one obtains these Clifford algebras as tensor products of cyclic algebras. More generally, suppose f_1 and f_2 are forms of degree d in disjoint sets of variables, and let C_i be a generalized Clifford algebra for f_i $(i = 1, 2)$. We assume that k contains a d-th primitive root ξ of unity. Then we define the $\mathbf{Z}/d\mathbf{Z}$-graded tensor product $C_1 \widehat{\otimes} C_2$ as the ordinary tensor product equipped with the multiplication defined by $(a \otimes b)(c \otimes d) = \xi^{(\deg b)(\deg c)} ac \otimes bd$ for homogeneous elements $b \in C_2$ and $c \in C_1$. It turns out (theorem 3.1) that $C_1 \widehat{\otimes} C_2$ is a generalized Clifford algebra for $f_1 + f_2$. Now if $f = a_1 X_1^d + \ldots a_n X_n^d$ is a diagonal form with $a_i \in k$, $a_i \neq 0$ for $i = 1, \ldots, n$, then $C_i = k[e_i]/(e_i^d - a_i)$ is a generalized Clifford algebra for $a_i X_i^d$, whence $C(f, \xi) = C_1 \widehat{\otimes} \ldots \widehat{\otimes} C_n$ is a generalized Clifford algebra for f, whose dimension over k is d^n. The structure of this algebra can be described quite easily. In theorem 3.6 it is shown that $C_0(f, \xi)$ is simple if n is odd and semisimple if n is even. The consequences for matrix factorizations of diagonal forms are formulated in theorem 3.9. At the end of this section we work out explicit factorizations of $\sum_{i=1}^{n} X_i^d$ over C.

Finally, in section 4 we show that a linear matrix factorization $f = \alpha_0 \cdot \ldots \cdot \alpha_{d-1}$ corresponds to a free module F over the hypersurface ring $R = S/(f)$ together with a filtration of F, whose quotients are linear MCM-modules over R. In particular, together with the results of section 3, it follows that a hypersurface ring of a diagonal form admits linear MCM-modules.

Many questions remain open [but see the STOP PRESS!]. We list a few of them:

1) Does every (homogeneous) form admit a finite-dimensional generalized Clifford algebra?
2) Do the linear MCM-modules together with R generate the Grothendieck group of R?
3) Can the periodicity theorem of Knörrer [K] be generalized to forms of higher degree?
4) Which forms can be transformed into diagonal forms?

We wish to thank T. G. Ivanova with whom we had many stimulating and helpful discussions, and P. M. Cohn for his valuable comments and suggestions. We also thank Bokut who informed us that L'vov and Nesterenko (answering a question of Krendelev) reported on the solution of question 1 at the 17:th All Union Algebra Conference in Minsk 1983, and announced this and related results (without proofs) in the Proceedings of that conference (pp 118 and 137, in Russian).

In particular, we thank the referee for putting our attention to the extensive work already done concerning generalized Clifford algebras (e.g. in [C], [H], [L], and [R]).

Finally we would like to express our gratitude to the organizers of the Fifth National School in Algebra in Varna, who brought together two of the authors of this paper and made possible many fruitful discussions with other participants of this conference that were indispensable for writing this paper.

1. Matrix factorizations and Clifford algebras

Let $f \neq 0$ be a homogeneous polynomial of degree $d \geq 2$ in the indeterminates $X_1, \ldots, X_n$ with coefficients in a field k.

DEFINITION 1.1. A (*linear*) *matrix factorization* of f (of size m) is an equation $f = \alpha_0 \cdot \ldots \cdot \alpha_{d-1}$, where the α_i are square matrices (of size m), whose entries are linear forms in the indeterminates $X_1, \ldots, X_n$ with coefficients in k, and f simply stands for f times the unit matrix E of size m.

We allow m to be infinite. In that case, however, we require that each row of the matrices has only finitely many nonzero entries, whence their products are defined, and that the product of any cyclic permutation of the matrices α_i is f again.

Given a matrix factorization $f = \alpha_0 \cdot \ldots \cdot \alpha_{d-1}$ and a $j \in \mathbf{Z}$, we set $\alpha_j := \alpha_i$, where $0 \leq i \leq d-1$ and $i \equiv j \pmod d$.Then, since any cyclic permutation of the factors again yields f as their product, it follows that $f = \alpha_i \cdot \alpha_{i+1} \cdot \ldots \cdot \alpha_{i+d-1}$ also is a matrix factorization for all $i \in \mathbf{Z}$.

Two matrix factorizations $f = \alpha_0 \cdot \ldots \cdot \alpha_{d-1}$, $f = \beta_0 \cdot \ldots \cdot \beta_{d-1}$ of the same size are called *equivalent* if there exists matrices $S_j \in \mathrm{Gl}(m,k)$ such that $\beta_j = S_j \alpha_j S_{j+1}^{-1}$ for all j.

The *sum* of the matrix factorizations $f = \alpha_0 \cdot \ldots \cdot \alpha_{d-1}$ and $f = \beta_0 \cdot \ldots \cdot \beta_{d-1}$ is the matrix factorization $f = \gamma_0 \cdot \ldots \cdot \gamma_{d-1}$, where

$$\gamma_i = \begin{pmatrix} \alpha_i & 0 \\ 0 & \beta_i \end{pmatrix}$$

for all i.

The matrix factorization $f = \alpha_0 \cdot \ldots \cdot \alpha_{d-1}$ is called *indecomposable* if it is not equivalent to a sum of matrix factorizations of f.

We consider the k-vector space $\bigoplus_{i=1}^{n} kX_i$ as the dual space of $V := \bigoplus_{i=1}^{n} ke_i$, where the basis $X_1, \ldots, X_n$ is dual to the basis $e_1, \ldots, e_n$.

Recall that a matrix α of size m with linear forms in n variables may be interpreted as a k-linear map $\phi\colon V \to \mathrm{Hom}_k(V_1, V_2)$, where V_1 and V_2 are m-dimensional k-vectorspaces (with specified bases): given a matrix α of linear forms and $x \in V$, we let $\alpha(x)$ be the matrix with coefficients in k, which is obtained from α by evaluating the entries of α at x. With respect to the given bases of V_1 and V_2 $\alpha(x)$ defines a linear map $\phi(x)\colon V_1 \to V_2$. We therefore may define

$$\begin{array}{ccc} V & \longrightarrow & \mathrm{Hom}_k(V_1, V_2) \\ x & \longmapsto & \phi(x) \end{array} .$$

Similarly one associates with $\phi\colon V \to \mathrm{Hom}_k(V_1, V_2)$ a matrix α of linear forms.

Therefore, given a matrix factorization $f = \alpha_0 \cdot \ldots \cdot \alpha_{d-1}$, there exist k-vector spaces $V_0 = V_d, V_1, \ldots, V_{d-1}$ such that α_i yields a linear map

$$\begin{array}{ccc} V & \longrightarrow & \mathrm{Hom}_k(V_i, V_{i+1}) \\ x & \longmapsto & \phi_i(x) \end{array} .$$

If we set $f(x) := f(x_1, \ldots, x_n)$ for $x = \sum x_i e_i \in V$, we obtain $f(x) \cdot \mathrm{id}_{V_0} = \phi_{d-1}(x) \circ \ldots \circ \phi_0(x)$ for all $x \in V$.

In this paper we are often dealing with $\mathbf{Z}/d\mathbf{Z}$-graded modules over $\mathbf{Z}/d\mathbf{Z}$-graded rings. If

$$M = \bigoplus_{i=0}^{d-1} M_i$$

is a $\mathbf{Z}/d\mathbf{Z}$-graded module and $j \in \mathbf{Z}$, we set $M_j = M_i$, where $0 \leq i \leq d-1$ and $i \equiv j \pmod{d}$. Then if we use the convention that $M(a)$ denotes the module shifted by a (so that $M(a)_i = M_{a+i}$ for all i), it follows that $M(a)$ is obtained from M by a cyclic permutation of the homogeneous components of M.

Given a matrix factorization $f = \alpha_0 \cdot \ldots \cdot \alpha_{d-1}$ of size m, we assign to it a $\mathbf{Z}/d\mathbf{Z}$-graded module M over the tensor algebra $T := T(V)$. We first choose a $\mathbf{Z}/d\mathbf{Z}$-graded k-vector space $M = \bigoplus_{i=0}^{d-1} M_i$, where $\dim_k M_i = m$ for all i.

Let $\phi_i \colon V \to \mathrm{Hom}_k(M_i, M_{i+1})$ be the k-linear maps associated with the matrices α_i, as described above. The T-module structure of M is then defined by the equation $x \cdot m = \phi_i(x)(m)$ for all $x \in V$, $m \in M_i$, and $i = 0, \ldots, d-1$.

Let $m \in M_i$ and $x \in V$. Then

$$(x^{\otimes d})m = \big(\phi_{i+d-1}(x) \circ \ldots \circ \phi_{i+1}(x) \circ \phi_i(x)\big)m = f(x)m$$

(where $x^{\otimes d} = \overbrace{x \otimes \ldots \otimes x}^{d \text{ copies}}$). It follows that the two-sided ideal $I(f) = \langle \{x^{\otimes d} - f(x) \mid x \in V\} \rangle$ is contained in the annihilator of M, so that M is a module over the $\mathbf{Z}/d\mathbf{Z}$-graded algebra

$$C(f) := T/I(f).$$

We call $C(f)$ the *universal (generalized) Clifford algebra* of f.

More generally we define (for $f \neq 0$)

DEFINITION 1.2. A *generalized Clifford algebra* for f is a $\mathbf{Z}/d\mathbf{Z}$-graded k-algebra C together with a monomorphism $V \hookrightarrow C_1$ of vector spaces such that
1) C is generated by V, and 2) $x^d = f(x)$ for all $x \in V$.

We shall see later (in theorem 1.8) that the universal Clifford algebra of f is indeed a generalized Clifford algebra for f. Then clearly $C(f)$ is universal in the sence that for any generalized Clifford algebra C for f there is a unique $\mathbf{Z}/d\mathbf{Z}$-graded epimorphism $\epsilon \colon C(f) \twoheadrightarrow C$ such that

$$\begin{array}{ccc} & \nearrow & C(f) \\ V & & \downarrow \epsilon \\ & \searrow & C \end{array}$$

commutes.

If f is a quadratic form, then $C(f)$ is the usual Clifford algebra.

If k is finite and $d \gg 0$, then we may pick an $f \neq 0$ of degree d, such that $f(x) = 0$ for *all* $x \in V$. In this 'pathological' case, $C(f)$ is a $\mathbf{Z}$-graded ring in the natural manner. On the other hand, if there is a $u \in V$ such that $f(u) = y \in k$, $y \neq 0$, then u is a *unit of degree* 1 in $C(f)$ (since $y^{-1}u^{d-1} \cdot u = y^{-1}f(u) = 1$). It is well-known that if k is infinite and $f \neq 0$ then f cannot act trivially on V; therefore we sometimes will demand k to be infinite, in order to ensure the existence of such a unit. (This is not a serious restriction, as remark 1.10 below shows.)

THEOREM 1.3. *Assume that k is infinite. Let $f \neq 0$ be a homogeneous polynomial of degree d.*

i. *The equivalence classes of matrix factorizations of f correspond bijectively to the isomorphism classes of $\mathbf{Z}/d\mathbf{Z}$-graded modules over the universal Clifford algebra of f.*

ii. *Let $M = \bigoplus_{i=0}^{d-1} M_i$ correspond to the matrix factorization $f = \alpha_0 \cdot \ldots \cdot \alpha_{d-1}$. Then*

1) $\dim_k M_i$ is equal to the size of the matrices α_i for all i.

2) If $j \in \mathbf{Z}$, then the shifted module $M(j)$ corresponds to the matrix factorization $f = \alpha_j \cdot \alpha_{j+1} \cdot \ldots \cdot \alpha_{j+d-1}$.

3) This matrix factorization is decomposable if and only if M is decomposable.

Proof. We just indicate how a $\mathbf{Z}/d\mathbf{Z}$-graded module $M = \bigoplus_{i=0}^{d-1} M_i$ defines a matrix factorization of f. Choose a $u \in V$ such that $f(u) \neq 0$; u is a unit in $C(f)$. Since $u \in C_1(f)$, the multiplication by u induces k-isomorphisms $u\colon M_i \xrightarrow{\sim} M_{i+1}$ for $i = 0, \cdots, d-1$, whence all M_i have the same k-vectorspace dimension. This implies that the k-linear maps $\phi_i\colon V \to \mathrm{Hom}_k(M_i, M_{i+1})$ for $i = 0, \cdots, d-1$ define square matrices α_i of linear forms (with respect to some bases of the M_i). Clearly $f = \alpha_0 \cdot \ldots \cdot \alpha_{d-1}$. □

We now describe the algebra $C(f)$ more precisely: For any $\ell \geq 0$, let N_ℓ be the set of n-tuples $\nu = (\nu_1, \ldots, \nu_n)$ with $\nu_i \geq 0$ for $i = 1, \ldots, n$ and $\sum_{i=1}^n \nu_i = \ell$. Let $N = \bigcup_{\ell \geq 0} N_\ell$. As usual, we set $x^\nu := x_1^{\nu_1} \cdot \ldots \cdot x_n^{\nu_n}$. Then (for some $a_\nu \in k$) we have

$$f = \sum_{\nu \in N_d} a_\nu x^\nu .$$

Let $\nu \in N$; a monomial in the generators $e_1, \ldots, e_n$ is said to have *multidegree* ν, if e_i occurs exactly ν_i times as a factor in this monomial for $i = 1, \ldots, n$. For example, the monomials of multidegree $(2,1)$ are $e_1^2 e_2$, $e_1 e_2 e_1$, and $e_2 e_1^2$.

We let g_ν be the sum of all monomials of multidegree ν, so that for instance $g_{(2,1)} = e_1^2 e_2 + e_1 e_2 e_1 + e_2 e_1^2$. For convenience we put $g_\nu = 0$ if ν is an n-tuple *not in* N, so that for instance $g_{(4,-1)} = 0$.

Let $J(f)$ be the two-sided ideal of T generated by the elements $g_\nu - a_\nu$, $\nu \in N_d$, and let $S(f) = T/J(f)$. Then we have

LEMMA 1.4.

i. $I(f) \subseteq J(f)$

ii. $I(f) = J(f)$, *if k is infinite.*

Proof. If $x = \sum_{i=1}^n x_i e_i \in V$, then

$$x^{\otimes d} - f(x_1, \ldots, x_n) = \sum_{\nu \in N_d} (g_\nu - a_\nu) x^\nu . \qquad \square$$

In other words, there is a natural epimorphism $C(f) \twoheadrightarrow S(f)$, which is an isomorphism if k is infinite.

Next we shall employ the "Diamond lemma" techniques (cf [Be]), in order to study the ideal $J(f)$.

If we set $e_1 < e_2 < \ldots < e_n$, we can order the monomials of T in the e_i in the standard way: first by length, then (for monomials of the same length) lexicographically.

Let $g \in T$ be an arbitrary non-zero element in the tensor algebra. g uniquely is a linear combination of monomials with non-zero coefficients. We denote by g^* (the *leading monomial* of g) the highest monomial occurring in this linear combination. If $I \subset T$ is a two-sided ideal, then we let I^* (the *associated monomial ideal* to I) be the two-sided ideal which is generated by all g^*, $g \in I$.

A subset $S \subset I$ is called a *standard basis* (or a *Gröbner basis*) of I, if g^*, $g \in S$, generates I^*. Any standard basis of I is a basis of I as well. (In the terminology of [Be], a given basis S of I is standard iff the corresponding system of reductions has no unresolvable ambiguities; c.f. e.g. [Be, **5.3**].)

The importance of these notions results from the following well-known

LEMMA 1.5. *Let B be the set of all monomials of T not belonging to I^*. Then the residue classes of the elements of B form a k-vector space basis of T/I.* □

Hence given a standard basis of I one can easily describe a k-vector space basis of T/I.

THEOREM 1.6. *The basis $\{g_\nu - a_\nu \mid \nu \in N_d\}$ of $J(f)$ described in lemma 1.4 is a standard basis of $J(f)$.*

Proof. For $\nu = (\nu_1, \dots, \nu_n) \in N$, let $m_\nu = g_\nu^* = e_n^{\nu_n} \cdots e_1^{\nu_1}$ and let $h_\nu = g_\nu - m_\nu$. Then what we want to prove is that

$$J(f)^* = I^* := T(m_\nu)_{\nu \in N_d} T \tag{1}$$

Also note that m_ν, $\nu \in N$, are the *non-increasing* monomials in $e_1, \dots, e_n$, i.e. the 'words' $e_{i_1} e_{i_2} \cdots e_{i_r}$ such that $j < l \implies i_j \geq i_l$.

In the sequel we adopt the terminology of [Be, **1**].

The *system of reductions* corresponding to the alleged standard basis is $S = \{\sigma_\nu \mid \nu \in N_d\}$, where $\sigma_\nu = (m_\nu, a_\nu - h_\nu)$.

The *ambiguities* all are on the form

$$a = (\sigma_\mu, \sigma_\nu, A, B, C) \ , \quad \mu, \nu \in N_d \ ,$$

where A, B, and C are non-trivial monomials **not** in I^* such that $m_\mu = AB$ and $m_\nu = BC$. Since m_μ and m_ν are non-increasing, so are A, B, C, and their product, whence $ABC = m_\lambda$ for some $\lambda \in N_\ell$, $\ell = d + \text{length}\, A$. Clearly λ is determined by ℓ and by μ and ν, since m_μ and m_ν are a right factor and a left factor, respectively, of m_λ. Conversely, the whole ambiguity a above is determined by λ, whence we put $a_\lambda :=$ this a.

Note that not all (μ, ν, ℓ) give rise to ambiguities, but that there is an ambiguity a_λ for any $\lambda \in N_\ell$, $d + 1 \leq \ell \leq 2d - 1$. However, also note that if this $\ell > d + 1$, then there are non-trivial monomials D and E, and a $\rho \in N_d$, such that $m_\lambda = D m_\rho E$. Then (as an easy and well-known argument shows) the ambiguity m_λ indeed is resolvable. Thus the only remaining ambiguities to check are

$$a_\lambda = (\sigma_\mu, \sigma_\nu, e_i, m_\kappa, e_j) \, , \quad \lambda \in N_{d+1} \ ,$$

where i and j are the highest and the lowest non-vanishing index, repectively, of the n-tuple λ, and where $\mu \in N_d$, $\nu \in N_d$, or $\kappa \in N_{d-1}$ is obtained by subtracting 1 from the j-component, from the i-component, or from both the i-component and the j-component of λ, respectively.

By inspection it is clear that if exactly s of the variables $e_1, \ldots, e_n$ occur in the 'word' m_λ, then no other variables can occur in any image of m_λ under any finite sequence of reductions. Hence we may forget the other variables in our analysis, and thus actually assume that $\lambda_l \geq 1$ *for* $l = 1, \ldots, n$. In particular we get

$$m_\lambda = m_\mu e_1 = e_n m_\nu = e_n m_\kappa e_1 \ .$$

We must show that there is some common 'image under reduction' of the two 'branches' $b_1 = r_{\sigma_\mu e_1}(m_\lambda)$ and $b_2 = r_{e_n \sigma_\nu}(m_\lambda)$. Recall that by definition b_1 is obtained by replacing m_μ by $a_\mu - h_\mu$ in m_λ, and similarly for b_2. Thus

$$b_1 = a_\mu e_1 - h_\mu e_1$$

and

$$b_2 = a_\nu e_n - e_n h_\nu \quad .$$

As a starter, let us note that in the case $n = 1$ we have $\mu = \nu = (d)$, and $b_1 = a_{(d)} e_1 = b_2$, and we are through.

Next, assume that $n > 1$. For any $i, j = 1, \ldots, n$ and any $\rho = (\rho_1, \ldots, \rho_n) \in N$, let

$$\rho(i) = (\rho_1, \ldots, \rho_i - 1, \ldots, \rho_n)$$

and

$$\rho(i,j) = (\rho(i))(j) \ ;$$

thus e.g. $\mu = \lambda(1)$, $\nu = \lambda(n)$, and $\kappa = \mu(n) = \nu(1) = \lambda(1, n)$. Let us write $\lambda = (\lambda_1, \ldots, \lambda_n)$. We distinguish four cases, depending on whether $\lambda_1 = 1$ or $\lambda_1 > 1$, and on whether $\lambda_n = 1$ or $\lambda_n > 1$.

The case $\lambda_1 > 1$, $\lambda_n > 1$: In this case we have

$$g_\mu = \sum_{i=1}^{n} e_i g_{\mu(i)} = \sum_{i=1}^{n} e_i g_{\lambda(1,i)}$$

whence

$$b_1 = a_{\lambda(1)} e_1 - e_n h_{\lambda(1,n)} e_1 - \sum_{i=1}^{n-1} e_i g_{\lambda(1,i)} e_1 \quad .$$

We may reduce the leading terms $m_{\lambda(i)}$ in $g_{\lambda(1,i)} e_1$ for $i = 1, \ldots, n-1$:

$$\begin{aligned} r_{e_1 \sigma_{\lambda(1)}} \cdots r_{e_{n-1} \sigma_{\lambda(n-1)}}(b_1) &= a_{\lambda(1)} e_1 - e_n h_{\lambda(1,n)} e_1 - \sum_{i=1}^{n-1} e_i \Big(a_{\lambda(i)} - \sum_{j=2}^{n} g_{\lambda(i,j)} e_j\Big) \\ &= -\sum_{i=2}^{n-1} a_{\lambda(i)} e_i - e_n h_\kappa e_1 + \sum_{\substack{1 \leq i \leq n-1 \\ 2 \leq j \leq n}} e_i g_{\lambda(i,j)} e_j = \alpha \ , \end{aligned}$$

say. Similarly we get

$$b_2 = a_{\lambda(n)} e_n - e_n h_\kappa e_1 - \sum_{j=2}^{n} e_n g_{\lambda(j,n)} e_j$$

and

$$r_{\sigma_{\lambda(2)} e_2} \cdots r_{\sigma_{\lambda(n)} e_n}(b_2) = \alpha \ ,$$