Lecture Notes in Mathematics

I. Cherednik Ya. Markov
R. Howe G. Lusztig

Iwahori-Hecke Algebras and their Representation Theory

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Martina Franca, Italy 1999

Editors: M. Welleda Baldoni
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Lectures given at the C.I.M.E. Summer School
held in Martina Franca, Italy
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Preface

This text consists of notes of three courses given during the CIME summer school which took place in 1999 in Martina Franca, Italy. The subject was Iwahori-Hecke algebras and their representation theory. The program consisted of several courses taught by senior faculty and some advanced lectures given by young researchers.

The scheduled courses were

G. Heckman, *Representation theory of affine Hecke algebras*
R. Howe, *Affine-like Hecke algebras and p-adic representations theory*
G. Lusztig *Representations of affine Hecke algebras*
I.Cherednik *Hankel transform via double Hecke algebra*

The specialized lectures on more advanced topics were given by T. Haines, M. Nazarov, C. Kriloff, U. Kulkharni, K. Maktouf, J. Kim and G. Papadoupoulo.

The volume contains the notes of the courses by I. Cherednik, R. Howe, and G. Lusztig. G. Heckman was not able to provide notes for his course. We give some references later in the introduction.

In the remainder of the introduction we give some background material on affine Hecke algebras and extra references that complement the notes.

Two basic problems of representation theory are to classify irreducible representations and decompose representations occuring naturally in some other context. Algebras of Iwahori-Hecke type are one of the tools and were (probably) first considered in the context of representation theory of finite groups of Lie type. For example for $G = GL(2, \mathbb{F}_q)$, consider the question of decomposing the induced module

$$I = Ind_B^G[\mathbb{1}] := \{f : G \longrightarrow \mathbb{C} \ : \ f(gb) = f(g)\}, \qquad \iota(g)f(x) := f(g^{-1}x),$$

where B is the subgroup of upper triangular matrices. One is naturally led to consider the algebra of intertwining operators of the module I. These are linear endomorphisms $T : I \longrightarrow I$ satisfying $T \circ \iota(g) = \iota(g) \circ T$ for all $g \in G$.

This algebra of endomorphisms can be identified with the algebra of B-biinvariant functions on G with multiplication structure given by convolution. In the case of $GL(2, \mathbb{F}_q)$ it is generated (over $\mathbb{C}$) by an element T satisfying the relation $T^2 = (q-1)T + q$, and is called the finite Hecke algebra of type

A_1. Its generalization to type A_n is of independent interest to knot theory because it is a quotient of the braid group.

The Hecke algebras mentioned above play an important role in combinatorics and representation theory of $GL(n)$ and the symmetric group. An account of their role in the theory of finite groups of Lie type is detailed in [Car] and the references therein.

The above algebra is not what Hecke introduced in the context of automorphic forms. He defined certain operators $T(n)$ ($n \in \mathbb{N}$) that act on automorphic forms for congruence groups and studied their eigenvalues and relation to Dirichlet series. An introduction to this theory can be found in [Se]. When one considers automorphic forms in the adelic setting ([Ge], [Bump]) the operators $T(n)$ are very closely related to the previous example. One is led to consider the group of rational points $G(\mathbb{F})$ for a local field $\mathbb{F}$ with residual field of characteristic p and a compact open subgroup K. The operators $T(n)$ are K-biinvariant functions supported on certain cosets. This interpretation has led to a systematic study of the representation theory of groups over totally disconnected fields.

The talks of Howe gave an overview of this research area, particularly the role of these algebras. The notes are in the article of R. Howe and C. Krillof *Affine-like Hecke algebras and p-adic representation theory* The following is a brief list of the topics covered.

- Structure of p-adic groups and their associated affine Hecke algebras.
- Bruhat and Iwahori-Bruhat decompositions from a geometric perspective.
- Application of Hecke algebras in representations of the p-adic groups.

Lusztig's talks were focused on the special case when the compact open subgroup is an Iwahori subgroup. In this case very detailed knowledge of the representation theory is available due to his work partly joint with Kazhdan. His notes, G. Lusztig *Notes on affine Hecke algebras* cover the following topics:

- The affine Hecke algebra
- $\mathcal{H}$ and equivariant K-theory
- Convolution
- Subregular case

A very different field where Hecke algebras play a role is in the area of special functions. MacDonald has made various conjectures about the existence of orthogonal polynomials in several variables attached to root systems. These polynomials are related to the spherical functions of real and p-adic groups. The conjectures have a natural interpretation in the context of representations of Iwahori-Hecke algebras. Much of the work in this area *e.g.* by Cherednik, Heckman and Opdam make essential use of this structure. Heckmann's course provided an introduction to this area. We refer to the Bourbaki talks [He] and [M].

In Cherednik's notes the focus is on the advantages of the operator approach in the theory of Bessel functions and the classical Hankel transform. An account of these results can be found in the article I. Cherednik and Y. Markov, *Hankel transform via double Hecke algebra:*

- L-operator
- Hankel transform
- Dunkl operator
- Double H double prime
- Nonsymmetric eigenfunctions
- Inverse transform and Plancherel formula
- Truncated Bessel functions

We would like to thank the speakers and the participants for their scientific work which made the school a success.

The summer school brought together young researchers from many different countries with senior faculty. Funds were provided by C.I.M.E., the National Science Foundation and the European Community.

Welleda Baldoni and Dan Barbasch

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Table of Contents

Hankel transform via double Hecke algebra

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This paper is a part of the course delivered by the first author at UNC in 2000. The focus is on the advantages of the operator approach in the theory of Bessel functions and the classical Hankel transform. We start from scratch. The Bessel functions were a must for quite a few generations of mathematicians but not anymore. We mainly discuss the *master formula* expressing the Hankel transform of the product of the Bessel function by the Gaussian.

By the operator approach, we mean the usage of the Dunkl operator and the $\mathcal{H}''$, *double H double prime*, the rational degeneration of the double affine Hecke algebra. This includes the transfer from the symmetric theory to the nonsymmetric one, which is the key tool of the recent development in the theory of spherical and hypergeometric functions. In the lectures, the Hankel transform was preceded by the standard Fourier transform, which is of course nonsymmetric, and the Harish-Chandra transform, which is entirely symmetric.

We followed closely the notes of the lectures not yielding to the temptation of skipping elementary calculations. We do not discuss the history and generalizations. Let us give some references. The master formula is a particular case of that from [D]. Our proof is mainly borrowed from [C1] and [C2]. The nonsymmetric Hankel transform is due to C. Dunkl (see also [O,J]). We will see that it is equivalent to the symmetric one, as well as for the master formulas (see e.g. [L], Chapter 13.4.1, formula (9)). This is a special feature of the one-dimensional setup. Generally speaking, there is an implication nonsymmetric $\Rightarrow$ symmetric, but not otherwise.

We also study the *truncated Bessel functions*, which are necessary to treat negative half-integral k, when the eigenvalue of the Gaussian with respect to the Hankel transform is infinity. They correspond to the finite-dimensional representations of the double H double prime, which are completely described in the paper. We did not find proper references but it is unlikely that these functions never appeared before. They are very good to demonstrate the operator technique.

We thank D. Kazhdan and A. Varchenko, who stimulated the paper a great deal, M. Duflo for useful discussion, and CIME for the kind invitation.

1 L-operator

We begin with the classical operator

$$\mathcal{L} = (\frac{\partial}{\partial x})^2 + \frac{2k}{x}\frac{\partial}{\partial x}.$$

Upon the conjugation:

$$\mathcal{L} = |x|^{-k}\, \mathcal{H}\, |x|^k, \quad \mathcal{H} = (\frac{\partial}{\partial x})^2 + \frac{k(1-k)}{x^2}. \tag{1}$$

Here k is a complex number. Both operators are symmetric = even.

The φ-function is introduced as follows:

$$\mathcal{L}\varphi_\lambda(x,k) = 4\lambda^2\varphi_\lambda(x,k), \qquad \varphi_\lambda(x,k) = \varphi_\lambda(-x.k), \qquad \varphi_\lambda(0,k) = 1. \tag{2}$$

We will mainly write $\varphi_\lambda(x)$ instead of $\varphi_\lambda(x,k)$. Since $\mathcal{L}$ is a DO of second order, the eigenvalue problem has a two-dimensional space of solutions. The even ones form a one-dimensional subspace and the normalization condition fixes φ_λ uniquely. Indeed, the operator $\mathcal{L}$ preserves the space of even functions holomorphic at 0. The φ_λ can be of course constructed explicitly, without any references to the general theory of ODE.

We look for a solution in the form $\varphi_\lambda(x,k) = f(x\lambda, k)$. Set $x\lambda = t$. The resulting ODE is

$$\frac{d^2 f}{dt^2}(t) + 2k\frac{1}{t}\frac{df}{dt}(t) - 4f(t) = 0, \qquad \text{a Bessel-type equation.}$$

Its even normalized solution is given by the following series

$$\begin{aligned} f(t,k) &= \sum_{m=0}^{\infty} \frac{t^{2m}}{m!\,(k+1/2)\cdots(k-1/2+m)} \\ &= \Gamma(k+\frac{1}{2})\sum_{m=0}^{\infty}\frac{t^{2m}}{m!\Gamma(k+1/2+m)}. \end{aligned} \tag{3}$$

So

$$f(t,k) = \Gamma(k+\frac{1}{2})t^{-k+\frac{1}{2}}J_{k-\frac{1}{2}}(2it).$$

The existence and convergence is for all $t \in \mathbf{C}$ subject to the constraint:

$$k \neq -1/2 + n, \; n \in \mathbf{Z}_+. \tag{4}$$

The symmetry $\varphi_\lambda(x,k) = \varphi_x(\lambda,k)$ plays a very important role in the theory. Here it is immediate. In the multi-dimensional setup, it is a theorem.

Let us discuss other (nonsymmetric) solutions of (3) and (2). Looking for f in the form $t^\alpha(1+ct+\ldots)$ in a neighborhood of $t=0$, we get that the coefficients of the expansion

$$f(t) = t^{1-2k} \sum_{m=0}^{\infty} c_m t^{2m} \text{ at } t = 0$$

can be readily calculated from (3) and are well-defined for all k.

The convergence is easy to control. Generally speaking, such f are neither regular nor even. To be precise, we get even functions f regular at 0 when $k = -1/2-n$ for an integer $n \geq 0$, i.e. when (4) does not hold. These solutions cannot be normalized as above because they vanish at 0.

Note that we do not need nonsymmetric f and the corresponding $\varphi_\lambda(x) = f(x\lambda)$ in the paper. Only even normalized φ will be considered. The nonsymmetric ψ-functions discussed in the next sections are of different nature.

Lemma 1.1. *(a) Let $\mathcal{L}^\circ$ be the adjoint operator of $\mathcal{L}$ with respect to the* **C**-*valued scalar product $\langle f,g\rangle_0 = 2\int_0^{+\infty} f(x)g(x)dx$. Then $|x|^{-2k}\mathcal{L}^\circ|x|^{2k} = \mathcal{L}$.*
(b) Setting $\langle f,g\rangle = 2\int_0^{+\infty} f(x)g(x)x^{2k}dx$, the $\mathcal{L}$ is self-adjoint with respect to this scalar product, i.e. $\langle \mathcal{L}(f),g\rangle = \langle f,\mathcal{L}(g)\rangle$.

Proof. First, the operator multiplication by x is self-adjoint. Second, $(\frac{\partial}{\partial x})^\circ = -\frac{\partial}{\partial x}$ via integration by parts. Finally,

$$x^{-2k}\mathcal{L}^\circ x^{2k} = x^{-2k}((\frac{d^2}{dx^2})^\circ + (\frac{\partial}{\partial x})^\circ(\frac{2k}{x}))x^{2k} = x^{-2k}((\frac{\partial}{\partial x})^2 - \frac{\partial}{\partial x}(\frac{2k}{x}))x^{2k}$$
$$= x^{-2k}(x^{2k}(\frac{\partial}{\partial x})^2 + 4kx^{2k-1}\frac{\partial}{\partial x} + 2k(2k-1)x^{2k-2}$$
$$- 2kx^{2k-1}\frac{\partial}{\partial x} - 2k(2k-1)x^{2k-2}) = (\frac{\partial}{\partial x})^2 + \frac{2k}{x}\frac{\partial}{\partial x} = \mathcal{L}. \quad (5)$$

Therefore, $\langle \mathcal{L}(f),g\rangle = 2\int_0^\infty \mathcal{L}(f)gx^{2k}dx =$

$$2\int_{\mathbf{R}_+} f\mathcal{L}^\circ(x^{2k}g)\,dx = 2\int_{\mathbf{R}_+} fx^{2k}\mathcal{L}x^{-2k}(x^{2k}g)\,dx = \langle f,\mathcal{L}(g)\rangle. \quad (6)$$

Actually this calculation is not necessary if (1) is used. Indeed, $\mathcal{H}^\circ = \mathcal{H}$.

2 Hankel transform

Let us define the *symmetric Hankel transform* on the space of continuous functions f on **R** such that $\lim_{x\to\infty} f(x)e^{cx} = 0$ for any $c \in \mathbf{R}$. Provided (4),

$$(\mathbb{F}_k f)(\lambda) = \frac{2}{\Gamma(k+1/2)}\int_0^{+\infty} \varphi_\lambda(x,k)f(x)x^{2k}dx. \quad (7)$$

The growth condition makes the transform well-defined for all $\lambda \in \mathbf{C}$, because

$$\varphi_\lambda(x,k) \sim \mathrm{Const}(e^{2\lambda x} + e^{-2\lambda x}) \text{ at } x = \infty.$$

The latter is standard.

We switch from $\mathbb{F}$ on functions to the transform of the operators: $\mathbb{F}(A)(\mathbb{F}(f) = \mathbb{F}(A(f))$. Remark that the Hankel transform of the function is very much different from the transform of the corresponding multiplication operator. The key point of the operator technique is the following lemma.

Lemma 2.1. *Using the upper index to denote the variable (x or λ),*

$$(a)\quad \mathbb{F}(\mathcal{L}^x) = 4\lambda^2; \quad (b)\quad \mathbb{F}(4x^2) = \mathcal{L}^\lambda; \quad (c)\quad \mathbb{F}(4x\frac{\partial}{\partial x}) = -4\lambda\frac{d}{d\lambda} - 4 - 8k.$$

Proof. Claim (a) is a direct consequence of Lemma 1.1 (b) with $g(x) = \varphi_\lambda(x)$:

$$\mathbb{F}(\mathcal{L}f) = \langle\mathcal{L}f, \varphi_\lambda\rangle = \langle f, \mathcal{L}\varphi_\lambda\rangle = 4\lambda^2\langle f, \varphi_\lambda\rangle = 4\lambda^2\mathbb{F}(f)$$

. Claim (b) results directly from the $x \leftrightarrow \lambda$ symmetry of ϕ, namely, from the relation $\mathcal{L}^\lambda\varphi_\lambda(x) = 4x^2\psi_\lambda(x)$. Concerning (c), there are no reasons, generally speaking, to expect any simple Fourier transforms for the operators different from $\mathcal{L}$. However in this particular case: $[\mathcal{L}^x, x^2] = 4x\frac{\partial}{\partial x} + 2 + 4k$. Appling $\mathbb{F}$ to both sides and using (a), (b), $[4\lambda^2, \mathcal{L}^\lambda/4] = \mathbb{F}(4x\frac{\partial}{\partial x}) + 2 + 4k$. Finally

$$\mathbb{F}(4x\frac{\partial}{\partial x}) = -4\lambda\frac{d}{d\lambda} - 2 - 4k - 2 - 4k = -4\lambda\frac{d}{d\lambda} - 4 - 8k$$

.

Note that

$$[x\frac{\partial}{\partial x}, x^2] = 2x^2, \ [x\frac{\partial}{\partial x}, \mathcal{L}^x] = -2\mathcal{L}^x,$$

because operators $x\frac{\partial}{\partial x}, \mathcal{L}$ are homogeneous of degree 2 and -2. So $e = x^2$, $f = -\mathcal{L}^x/4$, and $h = x\frac{\partial}{\partial x} + k + 1/2 = [e, f]$ generate a representation of the Lie algebra $sl_2(\mathbf{C})$.

Theorem 2.1. (**Master Formula**) *Assuming that* $\mathrm{Re}\, k > -\frac{1}{2}$,

$$2\int_0^\infty \varphi_\lambda(x)\varphi_\mu(x)e^{-x^2}x^{2k}dx = \Gamma(k+\frac{1}{2})e^{\lambda^2+\mu^2}\varphi_\lambda(\mu), \tag{8}$$

$$2\int_0^\infty \varphi_\lambda(x)\exp(-\frac{\mathcal{L}}{4})(f(x))e^{-x^2}x^{2k}dx = \Gamma(k+\frac{1}{2})e^{\lambda^2}f(\lambda),$$

provided the existence of $\exp(-\frac{\mathcal{L}}{4})(f(x))$ *and the integral in the second formula.*

Proof. The left-hand side of the first formula equals $\Gamma(k+1/2)\mathbb{F}(e^{-x^2}\varphi_\mu(x))$. We set

$$\varphi_\mu^-(x) = e^{-x^2}\varphi_\mu(x), \ \varphi_\mu^+(x) = e^{x^2}\varphi_\mu(x).$$

They are eigenfunctions of the operators

$$\mathcal{L}_- = e^{-x^2} \circ \mathcal{L} \circ e^{x^2}, \ \mathcal{L}_+ = e^{x^2} \circ \mathcal{L} \circ e^{-x^2}.$$

To be more exact, $\varphi_\mu^\pm$ is a unique eigenfunction of $\mathcal{L}^\pm$ with eigenvalue 2μ, normalized by $\varphi_\mu^\pm(0) = 1$.

Express $\mathcal{L}_-$ in terms of the operators from the previous lemma.

$$\begin{aligned} e^{-x^2}(\frac{\partial}{\partial x})^2 e^{x^2} &= e^{-x^2}(e^{x^2}(\frac{\partial}{\partial x})^2 + 2(2x)e^{x^2}\frac{\partial}{\partial x} + (2+4x^2)e^{x^2}) \\ &= (\frac{\partial}{\partial x})^2 + 4x\frac{\partial}{\partial x} + 2 + 4x^2, \\ e^{-x^2}\frac{2k}{x}\frac{\partial}{\partial x}e^{x^2} &= e^{-x^2}(e^{x^2}\frac{2k}{x}\frac{\partial}{\partial x} + 2xe^{x^2}\frac{2k}{x}) = \frac{2k}{x}\frac{\partial}{\partial x} + 4k, \\ \mathcal{L}_- &= e^{-x^2}((\frac{\partial}{\partial x})^2 + \frac{2k}{x}\frac{\partial}{\partial x})e^{x^2} = \mathcal{L} + 4x\frac{\partial}{\partial x} + 2 + 4k + 4x^2. \end{aligned} \quad (9)$$

Analogously, $\mathcal{L}_+ = \mathcal{L} - 4x\frac{\partial}{\partial x} - 2 - 4k + 4x^2$. Now we may use Lemma 2.1:

$$\begin{aligned} \mathbb{F}(\mathcal{L}_-^x) =& \mathbb{F}(\mathcal{L}^x) + \mathbb{F}(4x^2) + \mathbb{F}(4x\frac{\partial}{\partial x}) + \mathbb{F}(2+4k) = \\ =& 4\lambda^2 + \mathcal{L}^\lambda - 4\lambda\frac{d}{d\lambda} - 4 - 8k + 2 + 4k = \mathcal{L}_+^\lambda. \end{aligned} \quad (10)$$

Thus

$$L_+^\lambda(\mathbb{F}\varphi_\mu^-) = \mathbb{F}(\mathcal{L}_-^x)(\mathbb{F}\varphi_\mu^-) = \mathbb{F}(\mathcal{L}_-^x\varphi_\mu^-) = 2\mu\mathbb{F}(\varphi_\mu^-),$$

i.e. $\mathbb{F}\varphi_\mu^-$ is an eigenfunction of $\mathcal{L}_+$ with the eigenvalue 2μ. Using the uniqueness, we conclude that $\mathbb{F}(\varphi_\mu^-)(\lambda) = C(\mu)e^{\mu^2}\varphi_\mu^+(\lambda)$ for a constant $C(\mu)$. However the left-hand side of the master formula is $\lambda \leftrightarrow \mu$ symmetric as well as $e^{\mu^2}\varphi_\mu^+(\lambda) = e^{\lambda^2+\mu^2}\varphi_\mu(\lambda)$. So $C(\mu) = C(\lambda) = C$. Setting $\lambda = 0 = \mu$, we get the desired.

The second formula follows from the first for $f(x) = \varphi_\mu(x,k)$. Move $\exp(\mu^2)$ to the left to see this. It is linear in terms of $f(x)$ and holds for finite linear combinations of φ and infinite ones provided the convergence. So it is valid for any reasonable f. We skip the detail.

3 Dunkl operator

The above proof is straightforward. One needs the self-duality of the Hankel transform and the the commutator representation for $x\,\partial/\partial x$. The self-duality holds in the general multi-dimensional theory. The second property is more special. Also our proof does not clarify why the master formula is so simple. There is a "one-line" proof of this important formula, which can be readily generalized. It involves the *Dunkl operator:*

$$\mathcal{D} = \frac{\partial}{\partial x} - \frac{k}{x}(s-1), \qquad \text{where } s \text{ is the reflection } s(f(x)) = f(-x). \quad (11)$$