Group Theory and General Relativity

REPRESENTATIONS OF THE LORENTZ GROUP AND THEIR APPLICATIONS TO THE GRAVITATIONAL FIELD

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INTRODUCTION

The use of group theory in physics became very widespread in the last two decades. The reasons are not hard to find. Firstly, the theory happens to be a natural mathematical language suitable for physical concepts to be expressed in. Secondly, the emergence of very complicated mathematical equations in physics that needed simplification in any possible way, and symmetry in physics is just one such aspect that can be used in simplification.

The first factor can easily be seen in quantum field theory. The description of state functions is made by means of a vector in a Hilbert space. On the other hand, the theory requires Poincaré symmetry, or invariance. But representation theory of the Poincaré group provides us with exactly this kind of mathematical tool that combines special relativity and quantum mechanics. A more sophisticated use of representations of the Poincaré group is subsequently achieved by associating a particle with each representation. This then leads to Wigner's famous classification of particles according to their spins and masses [E. P. Wigner, On Unitary Representations of the Inhomogeneous Lorentz Group, *Ann. Math.* **40,** 149 (1939)]. A further step leads to the classification of all invariant equations according to the representation. This leads to the very important result of finding the equation of motion associated with each representation. In this way one obtains the Dirac equation, the Proca equation, etc. [E. P. Wigner, Invariant Quantum Mechanical Equations of Motion, in *Theoretical Physics*, International Atomic Agency, Vienna, 1963, pp. 59–82].

Wigner's ideas changed our conceptional approach to physics. An example is provided by the use of compact groups. such as SU_2, SU_3, etc., in strong and weak interactions. Once again the theory of groups is mystically fit to describe mathematically the particle's quantum numbers, such as isopin, strangeness, etc.

Although many excellent books on general relativity have been written, the present book is the first book on the subject of group theory and general relativity and deals with the applications of group theory to general relativity. As is well known, the theory of general relativity was founded at a time when group theory was very little used in physics. During the last two decades it has become obvious that general relativity is one of many theories known as gauge theories, and at the same time a lot of work has been done in general relativity using group theory. To be sure we do not have any other gauge theory (except for the obvious case of the electromagnetic theory) that is compared, in its perfection and fitness to experimental results, to general relativity. However, the idea of gauge theories in particle physics is very widespread. An example of this is Weinberg's gauge theory that combines the electromagnetic and weak interactions [S. Weinberg, *Phys. Rev. Letters* **19,** 1264 (1967)]. In this book an extensive discussion on the theory of general relativity from the point of view of gauge fields is given, and an attempt is made to put together in one volume many scattered original works on the use of group theory in general relativity theory. The point of view of looking at gravitation theory as a gauge theory was extensively discussed by the author in the NATO Advanced Study Institute on Mathematical Physics [M. Carmeli, $SL(2, C)$ Symmetry of the Gravitational Field, in *Group Theory in Non-Linear Problems*, A. O. Barut, Ed., D. Reidel, Dordrecht, Holland; Boston, U.S.A., 1974, pp. 59–110]. However, there is no other volume that encompasses the original articles, now scattered in the professional literature, which fits into the subject category of group theory and general relativity.

This book is based on lectures given by the author in the last four years to advanced undergraduate and graduate students of mathematics and physics at the Ben Gurion University. There are twelve chapters, divided into forty-six sections, five appendices, and an extensive bibliography. Each chapter concludes with a set of problems. The first six chapters are devoted to the theory of representations of the rotation and Lorentz groups. The other six chapters deal with the application of groups, mostly the Lorentz group, to the theory of general relativity. They cover topics that start from the fundamentals of general relativity and end with exact solutions of the gravitational field equations and representations of the Bondi–Metzner–Sachs group. No discussion on cosmology is included. Also, the chapter on the representations of the Bondi–Metzner–Sachs group is just a brief introduction to the subject. A more detailed account of this important group would need the use of the theory of representations of the Poincaré group, in particular Wigner's little group method; that was not the purpose of the present book. [The reader who is interested in more detail about the Bondi–Metzner–Sachs group is referred to R. Penrose' lucid review: Relativistic Symmetry Groups, in *Group Theory in Non-Linear Problems*, A. O. Barut, Ed., D. Reidel, Dordrecht, Holland; Boston, U.S.A., 1974, pp. 1–58, although no discussion is given on the representations themselves.] The whole book is written in a self-contained way in both topics of group theory and general relativity theory. No prior knowledge of either subject by the reader is assumed. The book could be used as a textbook for a two-semester course for students of mathematics and physics at the graduate level, or for research purposes. Parts of the book could also be used as a basis for a one-semester course; for example the first six chapters can be used as a text for a one-semester course on the theory of representations of the rotation and Lorentz groups for advanced undergraduate and graduate students of mathematics and physics. As is well known, the theory of representations of the Lorentz group has traditionally been used as an introduction to the general theory of representations of groups. Another example is that the last six chapters of the book can be used as a text for graduate students of physics and mathematics on the theory of general relativity. The detail of the chapters is as follows:

Chapter 1 is devoted to the theory of representations of the rotation group. It includes such elementary concepts as the pure rotation group, the group SU_2, the very important concept of invariant integral over a group and, of course, the Wigner matrices of irreducible representations of the rotation group. While the rotation group has been widely covered in other texts, the parametrization of these representations here is not done through the traditional Euler angles but by other angles that describe rotations. In chapter 2 the discussion of the Lorentz group begins. This chapter includes an elementary discussion of the problem in general. Chapter 3 includes the important case of the finite-dimensional spinor representations of the Lorentz group. Here the group $SL(2, C)$ is introduced, and its relation to the Lorentz group is outlined. Chapters 4, 5, and 6 are devoted to the infinite-dimensional representations of the group $SL(2, C)$. These representations are the principal series, the complementary series, and the complete series.

The discussion of the infinite-dimensional representations starts in Chap. 4 by outlining the spaces of representations. These include several Hilbert spaces. Here also the theory of Fourier transform on the group SU_2 is introduced. The group operators are subsequently introduced, and the representation of the principal series is realized in these spaces. The complementary series is subsequently introduced in Chap. 5, where an operator formulation is also given. Chapter 6 then concludes the discussion of the infinite-dimensional representations. In this chapter some harmonic analysis of the group $SL(2, C)$ is also given.

The theory of general relativity first appears in Chap. 7 where the standard elements of the theory are given. Applications of the spinor representations to general relativity theory are given in Chap. 8. The Maxwell and Weyl spinors are introduced and classified accordingly. In Chap. 9 the general aspects of the theory of gauge fields are described. This includes the concept of isotopic spin and isotopic gauge transformations. Generalizations are then made to the Lorentz and Poincaré groups, and finally to the group $SL(2, C)$. This leads to our obtaining the gravitational field equations in the familiar form of Newman and Penrose. Thus our approach here is obtaining the Newman–Penrose equations for general relativity from gauge-theory principles. Later chapters of the book are devoted to solving the field equations of general relativity.

In Chap. 10 we analyse the gravitational field variables, proving the Goldberg–Sachs theorem, and dealing with choosing coordinate systems and asymptotic behaviour. In Chap. 11 we give exact solutions to the Newman–Penrose equations of general relativity. These include the Robinson–Trautman solution, the Newman–Tamburino solutions, the NUT–Taub solution, and all type D vacuum solutions, including the familiar Kerr solution. Finally, Chap. 12 concludes the text with the representations of the Bondi–Metzner–Sachs group. The five appendices give reviews of the theory of groups, reviews of the theories of finite and infinite-dimensional representations, whereas the last two appendices summarize the Newman–Penrose equations for the gravitational field of general relativity. The book is then concluded with the bibliography.

Although both the theory of representations of the Lorentz group and the theory of general relativity are presented here, and although the material includes reviews of some of the most recent developments in both topics, the present book does not cover all possible subjects on both topics. Among the following list of excellent books and monographs the remedies for some of these deficiencies can be found:

(1) B. L. van der Waerden, *Modern Algebra*, Fredric Ungar, New York, 1953.
(2) L. Pontrjagin, *Topological Groups*, Princeton University Press, New Jersey, U.S.A., 1946.
(3) E. P. Wigner, *Group Theory and its Application to the Quantum Mechanics of Atomic Spectra*, Academic Press, New York, 1959.
(4) M. A. Naimark, *Linear Representations of the Lorentz Group*, Pergamon Press, New York, 1964.
(5) I. M. Gelfand, M. I. Graev, and N. Ya. Vilenkin, *Generalized Functions*, Vol. 5: *Integral Geometry and Representation Theory*, Academic Press, New York, 1966.
(6) I. M. Gelfand, R. A. Minlos, and Z. Ya. Shapiro, *Representations of the Rotation and Lorentz Groups and their Applications*, Pergamon Press, New York, 1963.
(7) W. Rühl, *The Lorentz Group and Harmonic Analysis*, W. A. Benjamin, New York, 1970.
(8) A. Trautman, F. A. E. Pirani, and H. Bondi, *Lectures on General Relativity* (Brandeis 1964 Summer Institute on Theoretical Physics, Vol. 1), Prentice-Hall, Englewood Cliffs, N.J., U.S.A., 1965.
(9) J. L. Anderson, *Principles of Relativity Physics*, Academic Press, New York, 1967.
(10) W. R. Davis, *Classical Theory of Particles and Fields and the Theory of Relativity*, Gordon and Breach, New York, 1970.
(11) S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, John Wiley, New York, 1972.
(12) S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space–Time*, Cambridge University Press, Cambridge, England, 1973.
(13) C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation*, W. H. Freeman, San Francisco, 1973.
(14) A. Papapetrou, *Lectures on General Relativity*, D. Reidel, Dordrecht, Holland; Boston, U.S.A., 1974.

I would like to conclude this Introduction by citing a very relevant statement on the important role of group theory in physics, even though most researchers now realize and understand this role. However, it is a citation that has its own historical significance and is here very relevant in relation to gravitational theory, just as it was originally. It is a quotation from an introduction of A. Salam in a moment of deep insight, commenting on G. Racah's lectures on Lie groups given at the Institute for Advanced Studies at Princeton [A. Salam, The Formalism of Lie Groups, in *Theoretical Physics*, International Atomic Agency, Vienna, 1963, pp. 173–196]:

> Throughout the history of quantum theory, a battle has raged between the amateurs and professional group theorists. The amateurs have maintained that everything one needs in the theory of groups can be discovered by the light of nature provided one knows how to multiply two matrices. In support of this claim, they of course, justifiably, point to the successes of that prince of amateurs in this field, Dirac, particularly with the spinor representations of the Lorentz group. As an amateur myself, I strongly believe in the truth of the non-professionalist creed. I think perhaps there is not much one has to learn in the way of methodology from the group theorists except caution. But this does not mean one should not be aware of the riches which have been amassed over the course of years particularly in the most highly developed of all mathematical disciplines—the theory of Lie Groups. My lectures then are an amateur's attempt to gather some of the fascinating results for compact simple Lie groups which are likely to be of physical interest. I shall state theorems; and with a physicist's typical unconcern rarely, if ever, shall I prove them. Throughout, the emphasis will be to show the close similarity of these general groups with the most familiar of all groups, the group of rotations in three dimensions. In 1951 I had the good fortune to listen to Prof. Racah lecture on Lie groups at Princeton. After attending these lectures I thought this is really too hard; I cannot learn this; one is hardly ever likely to need all this complicated matter. I was completely wrong. Eleven years later the wheel has gone full cycle and it is my turn to lecture on this subject. I am sure many of you will feel after these lectures that all this is too damned hard and unphysical. The only thing I can say is: I do very much hope and wish you do not have to learn this beautiful theory eleven years too late.

*   *   *

Many people have helped me to prepare this book, from the first stages of writing to the final stage of reading it, partially or completely. I am in particular indebted to Professor S. Malin, without whose help, continuous encouragement, and reading of the manuscript, the book would probably have never been finished. I am indebted to Professor A. O. Barut whose kind invitation to the NATO Advanced Summer Institute in Istanbul gave me the opportunity to present the content of Chaps. 8 and 9 of the book in my two-week series of lectures there, and for his kind hospitality there. I am indebted to my teacher and colleague Professor Nathan Rosen, both for reading the manuscript and for his comments and encouraging remarks on it which I am sure has led to a better presentation of the material. Many thanks are also due to Professor L. Witten for both encouragement on the idea of the book and for critically reading and commenting on it, and to Professor G. Tauber and Dr. J. Bekenstein for comments and remarks. Last, but not least, I am indebted to my students M. Kaye and C. Charach for systematically reading the manuscript and for their comments and suggestions. Finally, I am grateful to Mrs. Y. Ahuvia for the excellent job she has made of typing the manuscript.

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CONTENTS

CHAPTER
ONE

THE ROTATION GROUP

In the following we find the irreducible representations of the three-dimensional pure rotation group, O_3. This is done by Weyl's method, which makes use of the homomorphism of the special unitary group of order two, SU_2, onto the rotation group. The representations are discussed in terms of two different parameterizations: (1) the angle of rotation in a specified direction and the spherical angles of the direction of rotation; and (2) the traditional Euler angles.

1-1 THE THREE-DIMENSIONAL PURE ROTATION GROUP

A linear transformation g of the variables x_1, x_2, and x_3, which leaves the form $x_1^2 + x_2^2 + x_3^2$ invariant, is called a *three-dimensional rotation*. The aggregate of all such linear transformations g provides a continuous group, which is formed from the set of all *real orthogonal* 3-dimensional matrices[1] and is called the *three-dimensional rotation group*. The determinant of every orthogonal matrix is equal to either $+1$, in which case the transformation describes *pure rotation*, or to -1, in which case it describes a *rotation-reflection*. The aggregate of all pure rotations forms a group, which is a subgroup of the 3-dimensional group, and is called the *pure rotation group*. *This chapter is concerned with the 3-dimensional pure rotation group*. This group is denoted by O_3.

The Euler Angles

Let g be an element of the group O_3, i.e., a 3-dimensional orthogonal matrix with determinant unity. One then can express each such element in terms of a set of three parameters. An example of such parameters is that of Euler angles, which are

[1] A matrix g is called orthogonal if $g'g = 1$, where g' is the transposed of g.

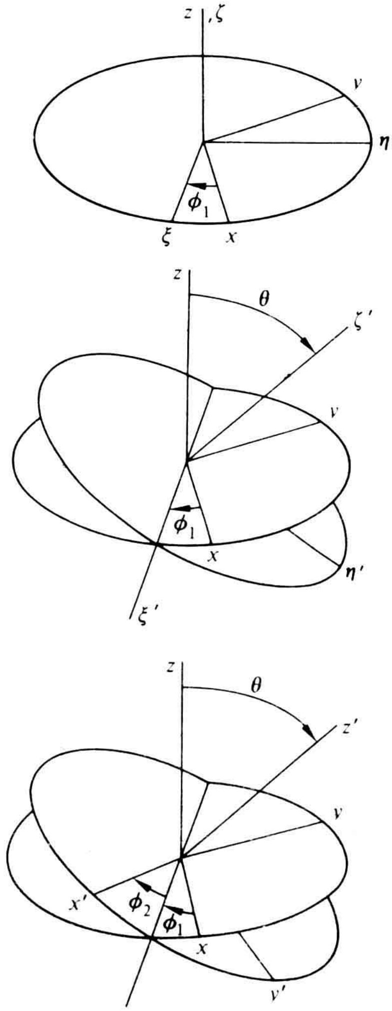

Figure 1.1 The three rotations defining the Euler angles.

defined as the three successive angles of rotation describing the transformation from a given Cartesian coordinate system to another by means of three successive rotations performed in a specific sequence.

The sequence will be started (see Fig. 1.1) by rotating the original system of axes $\mathbf{x}$ by an angle ϕ_1 clockwise about the z axis.[2] The new coordinate system will be labelled $\boldsymbol{\xi}$. One thus has $\boldsymbol{\xi} = g(\phi_1)\mathbf{x}$, where

$$g(\phi_1) = \begin{pmatrix} \cos\phi_1 & -\sin\phi_1 & 0 \\ \sin\phi_1 & \cos\phi_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1-1}$$

[2] We use the notation $\mathbf{x} = (x, y, z) = (x_1, x_2, x_3)$, $\boldsymbol{\xi} = (\xi, \eta, \zeta)$, $\boldsymbol{\xi}' = (\xi', \eta', \zeta')$, and $\mathbf{x}' = (x', y', z') = (x'_1, x'_2, x'_3)$.