GAME THEORY FOR ECONOMIC ANALYSIS

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Preface

The purpose of this book is to present the development over the past three decades of three strands in *n*-person game theory, with strong emphasis on applications to general economic equilibrium analysis. The first strand, represented by the *Nash equilibrium* of a game in normal form, studies an equilibrium concept of a society in which everybody behaves noncooperatively and passively. The second strand, represented by the *core* of a game in characteristic function form, studies the stable outcomes of a society in which everybody is aware of what he can do by cooperating with other members. The third strand, represented by the *Shapley value* of a side-payment game in characteristic function form, studies the fair outcomes of a society in which everybody is aware of what he can contribute to any group of members of the society by joining the group.

To capture the game-theoretical idea of each strand, several formulations with diverse degrees of generality have been proposed and studied in the past. A simple formulation conveys the essence of the idea in a straightforward manner, but it may not be general enough for useful economic applications. A complex formulation is general and powerful for economic applications, but beginners may find it hard to capture the essence. For each strand, therefore, I first present the game-theoretical idea in a simple setup and then gradually generalize it to more complex situations. This task is followed by discussion showing how the game-theoretical solution concept of each strand, generalized to the appropriate degree, can be applied to general economic equilibrium analysis. The model of a pure exchange economy is chosen, and three existence theorems for this model are established: the competitive equilibrium existence theorem (an application of a generalized Nash equilibrium existence theorem), the core allocation existence theorem (an application of a theorem for nonemptiness of the core), and the value allocation existence theorem (an application of a generalized Shapley value existence theorem). Having thus bridged *n*-person game theory and economic theory, I proceed to present several issues in mathematical economics dealing with the three economic concepts (competitive equilibrium, core allocation, and value allocation) within the framework of pure exchange economies; in particular, I present the fundamental theorems of welfare economics and limit theorems of cores and of value allocations. I also present a still more general game-theoretical concept, developed recently by myself: a social coalitional equilibrium. I do not discuss its economic applications here, however, because in order to do so I would have to give a disproportionately long discussion.

The text is organized as follows: Chapters 1-3 are devoted to some mathematical tools and theorems, which are usually not covered in standard mathematics courses but which play crucial roles in this text. In Chapter 4 (5, 6, respectively), a systematic account is given for the first (second, third, respectively) strand of n-person game theory mentioned above. Sections 4.4, 5.5, and 6.4 are the bridges between game theory and mathematical economics (a competitive equilibrium existence theorem, a core allocation existence theorem, and a value allocation existence theorem). Most of the later sections of the three chapters deal with mathematical economics. A social coalitional equilibrium is discussed in Sections 5.7 and 5.8. Sections 4.7, 5.9, and 6.6 are bibliographical notes. There, the evolution of relevant concepts is surveyed and more recent results are stated.

The reader is assumed to be familiar with junior-level real analysis and linear algebra. Every effort has been made to make the exposition self-contained, given this mathematical background.

The present work arose out of several courses that I have given since spring 1978 at Carnegie-Mellon University and at The University of Iowa. It was George J. Fix, head of the Mathematics Department, Carnegie-Mellon University, who first offered me an opportunity to develop a course in game theory and mathematical economics. Juan Jorge Schäffer of Carnegie-Mellon University shaped up my mathematical thinking, not only through our delightful collaboration in cooperative game theory, but also through innumerable discussions on mathematical science in general. While the first draft of the text was being typed in spring 1981, some of my colleagues at The University of Iowa, in particular Michael Balch and John Kennan, suggested some improvements in the text. Richard P. McLean of The University of Pennsylvania read the entire first draft and gave me many pieces of valuable information and thoughtful suggestions; indeed, most of the revisions I have made since then originated from his suggestions. My results that are included in the text were established in my research project, supported by the National Science Foundation Grant SES 8104387 (formerly, SOC 78-06123). I would like to thank the staff of Academic Press for the excellent work that was done to produce this book. To all the individuals and the institutions mentioned in this paragraph, I would like to express my deep gratitude. Needless to say, I am solely responsible for any possible deficiencies in this book.

Contents

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Preliminary Discussion

The basic mathematical notation to be used throughout this text is summarized in Section 0.1. Since the game theory presented in Chapters 4–6 serves as *the* mathematical foundation for economic analysis, *the* typical economic model is presented in Section 0.2: the model of a pure exchange economy. The reader is encouraged to keep this model in mind when going through Chapters 4–6.

0.1. Basic Notation

Given a set A,

$\#A :=$ the cardinality of A.

Given a positive integer k,

$\mathbf{R}^k :=$ k-dimensional Euclidean space;
$\mathbf{R}^k_+ :=$ the nonnegative orthant of $\mathbf{R}^k$;
$\mathbf{R} := \mathbf{R}^1$;
$\mathbf{R}_+ := \mathbf{R}^1_+$.

For any $x, y \in \mathbf{R}^k$,

$x_i :=$ the ith coordinate of x, $i = 1, \ldots, k$;
$x \cdot y := \sum_{i=1}^{k} x_i y_i \equiv$ the Euclidean inner product of x and y;
$\|x\| := \sqrt{x \cdot x} \equiv$ the Euclidean norm of x;
$x \geq y$ means $x_i \geq y_i$ for every $i = 1, \ldots, k$;
$x > y$ means $x \geq y$ and $x \neq y$;
$x \gg y$ means $x_i > y_i$ for every $i = 1, \ldots, k$.

For any subsets S, T of $\mathbf{R}^k$,

$\mathring{S} :=$ the interior of S in $\mathbf{R}^k$;
$\bar{S} :=$ the closure of S in $\mathbf{R}^k$;
$S + T := \{x + y \in \mathbf{R}^k \mid x \in S, y \in T\}$;
$S - T := \{x - y \in \mathbf{R}^k \mid x \in S, y \in T\}$.

The algebraic concepts $S + T$ and $S - T$ should not be confused with the set-theoretic concepts $S \cup T$ and $S \backslash T$. Abbreviations for some phrases:

iff   if and only if;
w.l.o.g.   without loss of generality;
□   the end of the proof.

A positive integer n will be interpreted as the number of *players* throughout Chapters 4–6. A set of players is called a *coalition*:

$N := \{1, 2, \ldots, n\}$ is interpreted as the set of players;
$\mathcal{N} := 2^N \backslash \{\varnothing\}$ is interpreted as the family of nonempty coalitions.

For every $j \in N$

$$e^j := (0, \ldots, \overset{j}{1}, \ldots, 0) \in \mathbf{R}^n.$$

For every $S \in \mathcal{N}$,

$\chi_S := \sum_{j \in S} e^j$;
$\Delta^S :=$ the convex hull of $\{e^j \mid j \in S\}$ (see Section 1.1).

Let X be a convex subset of $\mathbf{R}^k$, and let $f: X \to \mathbf{R}$ be a function. The function f is called *quasi-concave in* X if for every $r \in \mathbf{R}$ the set

$\{x \in X \mid f(x) \geq r\}$ is convex or, equivalently, if for any $x, y \in X$ and any t in the unit interval $[0, 1]$ it follows that $f(tx + (1 - t)y) \geq \min[f(x), f(y)]$. The function f is called *concave in* X if for any $x, y \in X$ and any $t \in [0, 1]$, $f(tx + (1 - t)y) \geq tf(x) + (1 - t)f(y)$. It is called *strictly concave in* X if strict inequality holds true in the last inequality whenever x and y are distinct and $0 < t < 1$.

0.2. Pure Exchange Economy

The model of a pure exchange economy with l types of commodities and m consumers is reviewed. A *commodity bundle* is a point x in $\mathbf{R}^l$; it describes the quantity x_h of each commodity $h = 1, \ldots, l$. The ith consumer is characterized by a triple $(X^i, \precsim_i, \omega^i)$ of his consumption set X^i, his preference relation $\precsim_i$, and his initial endowment vector ω^i. The *consumption set* X^i is a subset of $\mathbf{R}^l$ and is interpreted as the set of all commodity bundles with which he can physically survive; the set X^i characterizes the "physical needs" of consumer i. Suppose he chooses a bundle $x^i \in X^i$. If $x_h^i > 0$ ($x_h^i < 0$, resp.), then he demands (supplies, resp.) $|x_h^i|$ units of commodity h. The *preference relation* $\precsim_i$ is a binary relation on X^i. The statement $[x^i \precsim_i x^{i'}]$ is interpreted as: the commodity bundle $x^{i'}$ is at least as desirable as the commodity bundle x^i to consumer i. The relation $\precsim_i$, therefore, characterizes the "taste" of consumer i. Given $x^i, x^{i'} \in X^i$, denote by $[x^i >_i x^{i'}]$ the negation of $[x^i \precsim_i x^{i'}]$. Denote by $[x^i \sim_i x^{i'}]$ the statement $[x^i \precsim_i x^{i'}$, and $x^{i'} \precsim_i x^i]$, that is interpreted as: Consumer i is indifferent as to the choice of commodity bundle x^i or commodity bundle $x^{i'}$. The preference relation $\precsim_i$ is called *complete* if for any $x^i, x^{i'} \in X^i$ it follows that $x^i \precsim_i x^{i'}$ or $x^{i'} \precsim_i x^i$; completeness means that consumer i has a strong opinion on the commodity bundles. It is called *transitive* if for any $x^i, x^{i'}, x^{i''} \in X^i$ for which $x^i \precsim_i x^{i'}$ and $x^{i'} \precsim_i x^{i''}$ it follows that $x^i \precsim_i x^{i''}$; transitivity means that the consumer is rational. It is called *closed* if for any $x^i \in X^i$ the sets $\{\xi^i \in X^i \mid \xi^i \precsim_i x^i\}$ and $\{\xi^i \in X^i \mid x^i \precsim_i \xi^i\}$ are both closed in X^i; closedness means that his comparison of commodity bundles is smooth. It is called *weakly convex* if for any $x^i \in X^i$ the set $\{\xi^i \in X^i \mid x^i \precsim_i \xi^i\}$ is convex; weak convexity means diminishing marginal rate of substitution. It is called *convex* if for any x^i, $x^{i'} \in X^i$ for which $x^i >_i x^{i'}$ and for any real number t for which $0 < t < 1$ it follows that $tx^i + (1-t)x^{i'} >_i x^{i'}$. It is called *strictly convex* if for any two distinct x^i, $x^{i'} \in X^i$ for which $x^i \sim_i x^{i'}$ and for any real number t for which $0 < t < 1$ it follows that $tx^i + (1-t)x^{i'} >_i x^{i'}$. It is called *monotone* if for any x^i, $x^{i'} \in X^i$ for which $x^i > x^{i'}$ it follows that $x^i >_i x^{i'}$; monotonicity means that each commodity is desirable to consumer i. A commodity bundle $x^i \in X^i$ is called a *nonsatiation point* if there exists $x^{i'} \in X^i$ such that $x^{i'} >_i x^i$. A numerical function $u^i: X^i \to \mathbf{R}$ *represents* the preference relation $\lesssim_i$ if for any x^i, $x^{i'} \in X^i$, $x^i \lesssim_i x^{i'}$ if and only if $u^i(x^i) \leq u^i(x^{i'})$. A consumer whose preference relation is representable by a numerical function behaves *as though* he were trying to maximize this function. Sometimes such a numerical function is called a *utility function*. Proof of the following theorem (Theorem 0.2.1) can be found in Debreu (1959, Theorem (1), pp. 56–59).

Theorem 0.2.1. *Let X^i be the consumption set of consumer i, and let $\lesssim_i$ be his preference relation. Assume X^i is a connected subset of $\mathbf{R}^l$. Then there exists a continuous numerical function on X^i that represents $\lesssim_i$ if and only if $\lesssim_i$ is complete, transitive, and closed.*

The *initial endowment vector* ω^i is a point in $\mathbf{R}^l$ and is interpreted as the commodity bundle consumer i holds initially. A *pure exchange economy* is now characterized by a list of specified data, $\mathscr{E} := \{X^i, \lesssim_i, \omega^i\}_{i=1}^m$.

Besides the exogenous data $\mathscr{E}$, economists identify the behavioral pattern of the economic agents and the mechanism at the "meeting place" that coordinates their behavior. Consumer behavior in accordance with a given pattern leads to economic outcomes usually characterized by particular values of appropriate endogenous variables. Economic theorists formulate the outcomes in terms of a suitable solution concept and try to understand them by deducing the properties of the solution.

The solution concept with the greatest importance and longest history is the competitive equilibrium: each consumer i observes a price vector $p \in \mathbf{R}^l_+ \backslash \{\mathbf{0}\}$ in the market. His own budget set $\gamma^i(p, p \cdot \omega^i) :=$

$\{\xi^i \in X^i \mid p \cdot \xi^i \leq p \cdot \omega^i\}$ is therefore determined, and within this constraint he chooses individualistically the commodity bundle that satisfies him the most. An equilibrium price vector is then determined in the market so that the total demand cannot exceed the total supply. Let $L := \{1, \ldots, l\}$. By the price–wealth homogeneity, one may restrict the price vector domain to $\{p \in \mathbf{R}^l_+ \mid \sum_{h=1}^l p_h = 1\}$ that will be, by abuse of notation, denoted by Δ^L throughout this text. Thus the *competitive equilibrium* of a pure exchange economy $\mathscr{E}$ is a pair $((x^{i*})_{i=1}^m, p^*)$ of members of $\prod_{i=1}^m X^i$ and Δ^L such that

(1) x^{i*} is a maximal element of $\{\xi^i \in X^i \mid p^* \cdot \xi^i \leq p^* \cdot \omega^i\}$ with respect to $\precsim_i$ for every i; and
(2) $\sum_{i=1}^m x^{i*} \leq \sum_{i=1}^m \omega^i$.

It is a solution concept based on noncooperative behavior of the consumers and on the market mechanism. In Section 4.4, however, its existence problem will be discussed as a particular case of the existence problem of a certain noncooperative solution concept that does not specifically involve the market mechanism. Other solution concepts for $\mathscr{E}$ based on a cooperative behavior will be discussed in Sections 5.5 and 6.4.

Certain dynamic economies can be analyzed within the framework of the above static model $\mathscr{E}$. One characterizes a commodity not only by its physical properties and the place where it is available, but also by the date when it will be available and (in the case of uncertainty about the future) by the elementary event that will be realized. Commodity h, for example, is defined as coffee ice-cream that will be available in Iowa City in 77 days when it is snowing. The above definition of competitive equilibrium allows for this dynamic interpretation, but one should keep in mind that all futures markets (and also all contingent markets in the presence of uncertainty) are assumed to exist.

For the case $l = m = 2$ and $X^i = \mathbf{R}^2_+$ for $i = 1, 2$ the competitive equilibrium $((x^{i*})_{i=1,2}, p^*)$ is illustrated in the *Edgeworth box diagram* (see Figure 0.2.1). A commodity bundle of consumer 1 is measured from the origin $\mathbf{0}^1$ by the $\overrightarrow{\mathbf{0}^1 x_1^1}$ and $\overrightarrow{\mathbf{0}^1 x_2^1}$ axes. A commodity bundle of consumer 2 is measured from the origin $\mathbf{0}^2$ by the $\overrightarrow{\mathbf{0}^2 x_1^2}$ axis and the $\overrightarrow{\mathbf{0}^2 x_2^2}$ axis. The second origin $\mathbf{0}^2$ lies at the point $\omega^1 + \omega^2$ when measured

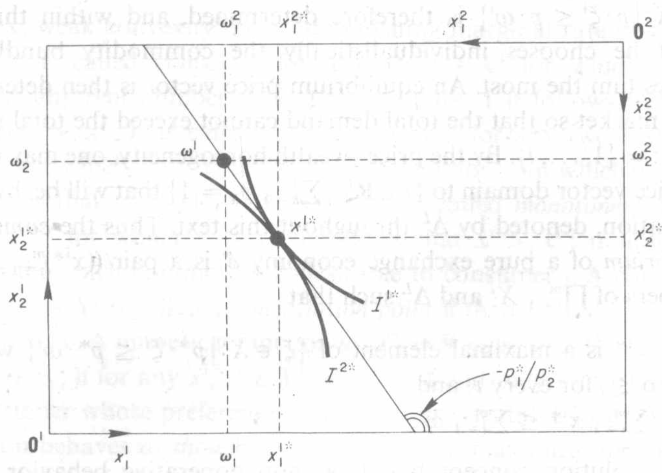

Figure 0.2.1 Edgeworth box diagram and competitive equilibrium.

from the origin $\mathbf{0}^1$ so it also represents the total supply vector. The curve I^{i*} is the indifference curve $\{x^i \in X^i \mid x^i \sim_i x^{i*}\}$ passing through x^{i*}, $i = 1, 2$. Now a competitive equilibrium is characterized by the following properties: (1) I^{1*} and I^{2*} are tangent at x^{1*} with the slope being $-p_1^*/p_2^*$; and (2) the tangent line passes through ω^1 so that it becomes the budget line.

1

Introduction to Convex Analysis

The basic ingredients of convex analysis are presented in the finite-dimensional setup. A definition of convexity, its immediate consequences, and related concepts are leisurely exposed in Sections 1.1–1.3. The deepest result is that of Theorem 1.4.4 of Section 1.4, in which some (Euclidean) topological concepts for convex sets are characterized in terms of vector space structure. In Section 1.5 two versions of the separation principle are established: the support theorem (Theorem 1.5.3) and the separation theorem (Theorem 1.5.4). Two other versions are also given in Exercises 6 and 7: the Hahn–Banach theorem and the subdifferentiability theorem. These four versions are equivalent in the sense that given any one of them the other three follow immediately. In Section 1.6 the concept of extreme point is introduced, and its elementary existence theorem (Theorem 1.6.1) is provided. Also introduced is a more general concept, the facial space. Artstein's fundamental lemma (Lemma 1.6.4) on facial spaces is presented, and its applications are discussed; in particular, the Shapley–Folkman theorem is proved. Assertions of certain theorems in this chapter are false in the infinite-dimensional context. Indeed, the negation of certain assertions characterizes infiniteness of the dimension of a given vector space. This last issue is discussed in the Appendix to this chapter. For general references pertinent to convex analysis see Fenchel (1951) and Rockaffellar (1970); also see Nikaidô (1968), which contains applications to economics.

1.1. Convex Set

A subset C of $\mathbf{R}^n$ is called *convex* if $[x, y \in C, \alpha \in \mathbf{R}, 0 \leq \alpha \leq 1]$ implies $\alpha x + (1 - \alpha)y \in C$. Given an arbitrary subset S of $\mathbf{R}^n$, one can associate with it naturally a convex set called the convex hull of S. Note that for an indexed family $\{C_i\}_{i \in I}$ of convex subsets of $\mathbf{R}^n$, $\bigcap_{i \in I} C_i$ is a convex set. The *convex hull* of S is the set $\text{co } S := \bigcap\{C \mid C$ is a convex subset of $\mathbf{R}^n$. $C \supset S\}$; it is the smallest convex set that contains S. The set co S is now characterized.

Let $(x^i)_{i \in F}$ be a finite set in $\mathbf{R}^n$. A point y in $\mathbf{R}^n$ is called a *convex combination* of $(x^i)_{i \in F}$ if there exists a nonnegative real coefficient α_i for each $i \in F$, with $\sum_{i \in F} \alpha_i = 1$, such that $y = \sum_{i \in F} \alpha_i x^i$. Let S' be the set of all convex combinations of finitely many members of S. By showing that any convex subset of $\mathbf{R}^n$ that contains S also contains S' and that S' itself is convex one can prove

Theorem 1.1.1. *The convex hull of a subset S of $\mathbf{R}^n$ is precisely the set of all convex combinations of finitely many members of S.*

Theorem 1.1.1 holds true for an arbitrary vector space over $\mathbf{R}$. A sharper result in the finite-dimensional context is

Theorem 1.1.2 (Carathéodory). *The convex hull of a subset S of $\mathbf{R}^n$ is precisely the set of all convex combinations of $(n + 1)$ members of S.*

Lemma 1.1.3. *Let $(x^i)_{i \in F}$ be a finite set in $\mathbf{R}^n$ and y be its nonnegative linear combination. Then there exists $F_0 \subset F$, with $\#F_0 \leq n$, such that y is a positive linear combination of $(x^i)_{i \in F_0}$.*

PROOF. Let $y = \sum_{i \in F} \alpha_i x^i$, and assume w.l.o.g. that $\alpha_i > 0$ for all $i \in F$.

Step I. If $(x^i)_{i \in F}$ is linearly dependent, then $\exists F' \subsetneqq F : y$ is a positive linear combination of $(x^i)_{i \in F'}$. Indeed, there exist γ_i, $i \in F$, not all 0, such that $\sum_{i \in F} \gamma_i x^i = \mathbf{0}$. Without loss of generality $\exists i \in F : \gamma_i > 0$ (otherwise, multiply by -1). Define $\theta := \min_{i:\gamma_i > 0} \alpha_i/\gamma_i$, and consider $y = \sum_{i\in F} (\alpha_i - \theta\gamma_i)x^i$.

Step 2. Repeat the procedure of Step 1 until a linearly independent subset $(x^i)_{i\in F_0}$ is obtained. By linear independence $\#F_0 \le n$. □

PROOF OF THEOREM 1.1.2. Given a finite set $(x^i)_{i\in F}$ in $\mathbf{R}^n$, a vector $y(\in\mathbf{R}^n)$ is a convex combination of $(x^i)_{i\in F}$ iff $(y, 1)$ is a nonnegative linear combination of $((x^i, 1))_{i\in F}$ [with the same coefficients]. So

$$y \in \operatorname{co} S \Leftrightarrow \Big[y = \sum_{i\in F} \alpha_i x^i,\ \#F < \infty,\ x^i \in S,\ \alpha_i \in \mathbf{R}_+,\ \sum_{i\in F} \alpha_i = 1\Big]$$

$$\Leftrightarrow \Big[(y, 1) = \sum_{i\in F} \alpha_i(x^i, 1),\ \#F < \infty,\ x^i \in S, \alpha_i \in \mathbf{R}_+\Big]$$

$$\Leftrightarrow \Big[(y, 1) = \sum_{i\in F_0} \beta_i(x^i, 1),\ \#F_0 \le n + 1,\ x^i \in S,\ \beta_i \in \mathbf{R}_+\Big]$$

$$\Leftrightarrow \Big[y = \sum_{i\in F_0} \beta_i x^i,\ \#F_0 \le n + 1,\ x^i \in S,\ \beta_i \in \mathbf{R}_+,\ \sum_{i\in F_0} \beta_i = 1\Big]. \quad \square$$

Corollary 1.1.4. *Let S be a subset of $\mathbf{R}^n$. If S is compact, then so is* co S.

PROOF. Denote by Δ the set $\{\alpha \in \mathbf{R}^{n+1} \mid \alpha \ge \mathbf{0},\ \sum_{i=1}^{n+1} \alpha_i = 1\}$, and define the function

$$f\colon \overbrace{S \times \cdots \times S}^{n+1} \times \Delta \to \mathbf{R}^n$$

by $f(x^1, \ldots, x^{n+1}, \alpha) := \sum_{i=1}^{n+1} \alpha_i x^i$. The function f is continuous on its compact domain $S \times \cdots \times S \times \Delta$, so its image $f(S \times \cdots \times S \times \Delta)$ is compact. But the image is precisely co S by Carathéodory's theorem (Theorem 1.1.2). □

A finite subset $(x^i)_{i\in F}$ of $\mathbf{R}^n$ is called *affinely independent* if

$$\Big[\sum_{i\in F} r_i x^i = \mathbf{0},\ r_i \in \mathbf{R},\ \sum_{i\in F} r_i = 0\Big]$$

implies $r_i = 0$ for each $i \in F$; or, equivalently, if with an arbitrarily chosen $i_0 \in F$ the set $(x^i - x^{i_0})_{i \in F \setminus \{i_0\}}$ is linearly independent. A subset S of $\mathbf{R}^n$ is called a *k-dimensional simplex* if there is an affinely independent set $(x^i)_{i \in F}$, with $\#F = k + 1$, such that $S = \text{co}(x^i)_{i \in F}$; those x^i are called the *vertices* of S. Each point of a simplex is uniquely expressed as a convex combination of the vertices. A simplex is compact and convex. Frequently one can easily prove theorems on a compact, convex set by first establishing the results on a simplex.

1.2. Affine Subspace

A class of convex sets plays a central role in analyzing algebraic and topological properties of (general) convex sets. A subset M of $\mathbf{R}^n$ is called an *affine subspace* if there exist a point m and a subspace W of $\mathbf{R}^n$ such that $M = \{m\} + W$. Given an affine subspace M, such a subspace W is uniquely determined. Indeed,

Lemma 1.2.1. *Let M be an affine subspace of $\mathbf{R}^n$, say $M = \{m\} + W$ for a point m and a subspace W of $\mathbf{R}^n$. Then $m \in M$ and $W = M - M$.*

PROOF. The first of the conclusions is straightforward since $\mathbf{0} \in W$. If $x \in W$, then $2x \in W$. So $m + x, m + 2x \in M$. Therefore $x = (m + 2x) - (m + x) \in M - M$. This proves $W \subset M - M$. Choose any $m^1, m^2 \in M$. There exist $x^i \in W$ such that $m^i = m + x^i$, $i = 1, 2$. Then $m^1 - m^2 = x^1 - x^2 \in W$. Thus $M - M \subset W$. □

The *dimension* of an affine subspace M is defined as the dimension of the unique subspace $(M - M)$. A characterization of an affine subspace is

Theorem 1.2.2. *Let M be a subset of $\mathbf{R}^n$. The set M is an affine subspace iff $[x, y \in M, r \in \mathbf{R}]$ implies $rx + (1 - r)y \in M$.*