PHYSICAL SIMILARITY AND DIMENSIONAL ANALYSIS

An Elementary Treatise

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PREFACE

I may fairly claim to have one qualification for writing a book on dimensional analysis and that is a keen and long-standing interest in the subject. This dates back to my student days when I was greatly impressed by the examples of dimensional analysis given in Poynting and Thomson's *Text Book of Physics*. However, my first publication on the subject was an article called " A Review of Dimensional Analysis " which appeared in *Engineering* during June 1949. This aroused some interest in this country and abroad and I felt encouraged to prepare an elementary book on the subject. The present volume is the result.

A considerable number of books on dimensional analysis have already appeared and the publication of another needs some justification. This justification is provided, I believe, by the fact that this is the first book in which dimensional analysis is discussed in its proper context, namely, the theory of physical similarity. The theory of similarity is treated with considerable thoroughness in this book, although at an elementary level. The point of view consistently adopted is that dimensional analysis is a particularly convenient technique for obtaining the quantitative conditions for similarity in behaviour of a family of similar physical systems and the consequences of this similarity in behaviour. Throughout the book I have used the phrase *measure formula* as a synonym for *physical dimensions* and I consider that " measure formula " is much to be preferred because it is a simple and correct description of the entity. There is a metaphysical smack about " physical dimensions " which has misled and bamboozled many. A minor innovation, first introduced in the article in *Engineering* already referred to, is the use of the special symbol $\triangleq$ to indicate dimensional equality, i.e. identity of measure formulae. I have been careful to avoid the abominable practice exemplified by using the symbol *Re* for Reynolds number, written as if it were the mathematical product of *R* and *e*.

The illustrative matter included in this book covers a fairly wide field, but there has been no attempt at exhaustiveness. A special chapter is devoted to the increasingly important applications of non-dimensional coefficients in engineering design.

I am indebted to Professor B. Hague for reading Chapter 9 and to Miss E. D. Wyper for her patient work in typing the manuscript.

Readers approaching the subject of the book for the first time are recommended to begin by reading Chapters 1, 2 and 3, followed by §§ 4.1 and 4.2 and Chapter 5. The remainder of the text can then be read in any order convenient to the reader.

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CHAPTER 1

INTRODUCTORY SURVEY

1.1. Aims of this Chapter. The purpose of this chapter is to give the reader a brief introduction to the subjects which are discussed later in detail and to acquaint him with the general outlook and method of the book. The main theme of the book is dimensional analysis and the aim is to discuss and exemplify this in its proper context, which is the theory of physical similarity.

It is opportune to say that only very elementary mathematics is employed in most of the text. The occasional excursions into "higher" mathematics are mainly in the examples, which the reader may omit without losing the thread of the argument.

1.2. Similar Physical Systems. The idea of similarity first acquired a precise meaning in geometry. In Euclidean geometry two plane figures are similar when corresponding angles are equal and when corresponding sides are in a constant ratio; such figures have the same shape but they may differ in size and in position. However, a physical body is more than a mere geometric figure as it has mass and other physical attributes. Hence the physical similarity of two bodies signifies more than the geometric similarity of their forms. For example, the dynamical behaviour of two bodies would only be similar, in general, if the densities at geometrically corresponding points were in a constant ratio. Similarly, two bodies could only be similar as regards the conduction of heat if their thermal conductivities at geometrically corresponding points were in a constant ratio. Accordingly we regard two bodies or systems as similar only when their relevant physical properties are similarly distributed. For further details the reader should consult § 3.5.

1.3. Similarity of Phenomena. The concept of similarity in phenomena applies both to the static and dynamic behaviour of physical systems. As a simple example of similarity in static behaviour let us consider a pair of similar beams similarly supported and loaded. Then the deflected forms of the beams will be geometrically similar when there is a certain relationship between the linear dimensions, elastic moduli and intensities of loading of the two beams. When this relationship is satisfied the deflexions of the beams at geometrically corresponding points will be in proportion to their linear dimensions.

When we pass on to consider similarity in unsteady or time-dependent phenomena we have to introduce the idea of *corresponding times*. These are such that the time intervals measured from suitable time origins or initial instants are in a fixed ratio. Consider now, as a simple example, a pair of particles A and B moving in a plane. We define their motions to be similar if it is possible to choose initial instants such that, at corresponding times, the corresponding displacements (as vectors) are in a fixed ratio. It readily follows from this that the paths of the particles are geometrically similar, but this by itself does not secure similarity of the motions. For example, we could have A and B describing the same curve but reaching corresponding points at times related by

$$t_A = kt_B^2,$$

and then the motions would not be similar. Whenever two motions are similar, the velocities at corresponding points of the paths are in proportion ; likewise the accelerations at corresponding points are also in proportion, and the same is true of the resultant applied forces. This illustrates the important point that when two physical systems are similar and behaving similarly, any knowledge about one system also provides knowledge of the other system.

It is of fundamental importance to recognize that the behaviour of two similar systems is not necessarily similar. Thus, in the example of beam deflexion already considered, the deflected forms of the similar beams were strictly similar only when a certain relationship was satisfied. The determination of the quantitative conditions for similarity of behaviour is an essential part of the study of physical similarity and these are most conveniently found by the technique of dimensional analysis. The same technique simultaneously establishes the consequences of similarity in behaviour (see § 1.5).

1.4. Units and Measures. In § 1.2 we have spoken of similar distributions of physical properties such as density and thermal conductivity. We cannot determine whether such distributions are similar or not until we have measured the quantities in question and this requires two things :

(*a*) the selection and establishment of a unit,

(*b*) a technique of measurement by which the measure of the quantity can be found.

The process of measurement consists in a direct or, more often, indirect comparison with the unit and the measure is the ratio of the physical quantity to the chosen unit. The measure is thus a number and is not itself a physical quantity.

1.5. Dimensional Analysis. Dimensional analysis is based on the fact that an equation between the measures of physical quantities is dimensionally homogeneous. The physical dimensions or measure formula of a physical quantity is an expression showing the dependence of the magnitude of the proper unit of that quantity on the magnitudes of the fundamental units. For instance, the measure formula of area is L^2, where L represents the measure of a new unit of length in terms of the original unit of length, for the new consistent unit of area, which is a square whose side contains L of the original units of length, contains L^2 of the original units of area. The principle of dimensional homogeneity states that every term in the equation has the same measure formula.

Dimensional analysis is to be regarded as a special and convenient technique for finding the quantitative conditions for similarity of behaviour together with the consequences of this similarity. It is important for the reader to understand that the conclusions drawn from dimensional analysis can always be established directly. It is the ease and convenience of the process of dimensional analysis which gives it its importance.

The result of a dimensional analysis can always be put in the form that certain *non-dimensional quantities* are functionally related. A non-dimensional quantity is one whose measure is independent of the sizes of the fundamental units; for example, the coefficient of solid friction μ is non-dimensional, for it is the ratio of two forces whose measures vary in proportion when the fundamental units are altered. Especially in engineering, we are interested in one particular physical quantity which is of special importance, e.g. a particular force which causes stresses in a body. By convention the non-dimensional quantity containing the measure of this quantity and which can be used to calculate it is usually called a *non-dimensional coefficient*, whereas the non dimensional quantities on which it depends are called *non-dimensional parameters*. The Reynolds number (see §§ 6.2 and 6.3) is a famous example of a non-dimensional parameter. Non-dimensional coefficients and parameters are of great utility in engineering design (see Chap. 11). Dimensional analysis establishes the *laws of comparison* which relate the results of experiments on models with those obtained with their full-scale prototypes.

One important application of dimensional analysis is to the planning of experiments (see § 5.6 and Chap. 10). The guidance provided by dimensional analysis permits the greatest amount of useful information to be obtained from a programme of experiments and results in great economy of effort. Another valuable application is to the checking of equations and formulae (see § 5.2).

Non-dimensional quantities are sometimes called *numerics*. This term is not used in the text, but the reader may adopt it if he pleases.

1.6. Summary. The idea of physical similarity is based on the quantitative correspondence of two or more physical systems. When systems are similar and behaving similarly, the quantitative relations for one can be derived from those holding for another. This is of particular value in relating measurements on a model with the corresponding quantities for its full-scale prototype. Dimensional analysis is a particularly convenient technique for establishing the quantitative conditions for similarity and the consequences of similarity.

Note on the Use of Functional Symbols.

Throughout this book general functional symbols, such as $f(\quad)$ and $F(\quad)$, merely stand for undetermined functions of their variable or variables. Thus the same functional symbol will represent different functions, according to the context.

CHAPTER 2

UNITS AND MEASURES

2.1. Units and Physical Measurement. A body of organized knowledge worthy of the name of science may exist even when techniques of measurement have not been evolved; it is then a branch of *natural history* in the sense of the phrase introduced by Kelvin. However, techniques of measurement have in fact been established throughout the field of physical science and nearly all physical experimentation consists of measurement. The processes of measurement are multifarious and their detailed discussion does not concern us here, but the result of a measurement is always a *number* which expresses the ratio of the physical quantity measured to the chosen unit of like physical nature. *Measurement thus involves firstly the selection and establishment of a unit and secondly the development of a process of numerical comparison with the unit.* Once the unit has been selected, the measure of a given physical quantity becomes a unique number.

As we shall see (§ 2.3), units are either *fundamental* or *derived.* The size of a fundamental unit is largely arbitrary, but in practice a fundamental unit is a historical product whose magnitude has remotely been based on human convenience. This is very clearly shown by the example of the *foot* as a unit of length. As their name indicates, derived units are not selected arbitrarily but are obtained by some definite process from the fundamental units.

By the *establishment* of a unit we mean the setting up of the unit as a constant physical standard* of such a nature and form that comparison with physical quantities of the same kind can be performed easily and, above all, accurately. It is to be understood that absolute accuracy is necessarily unattainable in physical measurement, but the fundamental standard must be such that the comparison of other quantities with it can be performed with the highest attainable accuracy. Absolute constancy of the standard is a theoretical requirement which can never be perfectly attained but, as physical knowledge increases, the constancy of standards, such as those of length and time, is gradually improved. To be of practical value a unit must be accepted by

* The standard may be a fraction or multiple of the unit.

those concerned with its use and, in fact, the fundamental units are legally established in civilized countries.

It cannot be too strongly emphasized that we never calculate with physical quantities but always with their measures, which are numbers. The symbols which appear in the mathematical equations related to a physical problem or process likewise represent numbers (measures, coefficients, indices), operations with numbers or equality. To avoid circumlocution it is usual to say that such and such a symbol " is " the physical quantity in question, but it should never be forgotten that the symbol represents the number which is the measure of the physical quantity in terms of a selected unit.

2.2. The Measure of a Physical Quantity and its Dependence on the Unit. Let the symbol Q represent some definite physical quantity, and let U represent a unit, necessarily of like physical nature, in terms of which Q can be measured. Then the *measure* q of Q when U is adopted as the unit is the *number* which expresses the ratio of Q to U. The relation between Q, U and q can be represented symbolically by

$$Q = qU, \quad . \quad . \quad . \quad . \quad (2.2,1)$$

where the symbol = indicates physical equality. It is to be noted that this is not an algebraic equation; in an algebraic equation the sign of equality is = and indicates the equality of the pair of numbers connected by the symbol. We shall always use the symbols = and = with these meanings.

Suppose now that we change the unit from U to U'. Since U' is necessarily of the same physical nature as U it can be measured in terms of U. Let k be the measure of U' with U as unit. Then we have

$$U' = kU. \quad . \quad . \quad . \quad . \quad (2.2,2)$$

Suppose also that q' is the measure of Q with U' as unit. Then

$$Q = q'U' \quad . \quad . \quad . \quad . \quad (2.2,3)$$

$$= q'kU \quad . \quad . \quad . \quad . \quad (2.2,4)$$

by (2.2,2). Equations (2.2,1) and (2.2,4) both express Q in terms of the unit U and, since the measure of Q with a given unit is unique, we must have

$$q = q'k. \quad . \quad . \quad . \quad . \quad (2.2,5)$$

Here we have used the symbol = since this equation indicates the equality of two numbers. The equation can be written alternatively

$$q' = \frac{q}{k}. \qquad \qquad (2.2,6)$$

Thus we see that when we alter the unit of measurement in the ratio $k : 1$ we concurrently change the measure of a given physical quantity in the ratio $1 : k$. This very simple fact is fundamental for dimensional analysis.

2.3. Fundamental and Derived Units. Let us begin by considering the selection of a unit of area and let us suppose that a unit of length has already been chosen. A *possible* unit of area is that of a square whose side is the unit of length and this is, in fact, the most convenient unit of area. Let us take a rectangle whose sides have the measures a and b, which we may here assume to be integers. Then we can divide the rectangle into $a\,b$ squares each of unit side and, if A is the measure of the area of the rectangle, we shall accordingly have

$$A = ab. \qquad \qquad (2.3,1)$$

The important point is that the measure of the area is simply the product of the measures of the sides. Now, if we had taken an arbitrary unit of area, we should have had to write

$$A = kab, \qquad \qquad (2.3,2)$$

where k is a numerical coefficient. As a concrete instance, if the unit of length is a foot and the unit of area is a square foot, then equation (2.3,1) is valid. But if the unit of area is changed to the square yard while the unit of length remains the foot we get

$$A = \tfrac{1}{9}ab, \qquad \qquad (2.3,3)$$

The square foot as a unit of area is said to be *consistent with* the foot as a unit of length. Also this unit of area is called a *derived unit* since it can be constructed from a knowledge of the unit of length, which is then regarded as being a *fundamental unit*.*

Consistent derived units sufficient to cover all the quantities arising in a particular science can be constructed from a set of fundamental units by making use of the physical relations between the quantities. The equations connecting the measures of the quantities are then free from *arbitrary* coefficients.† Thus the unit of volume is that of a cube whose side is the unit of length,

* It would be equally logical to take the unit of area as fundamental and to derive from it a consistent unit of length.

† Non-arbitrary numerical multipliers sometimes occur, e.g. in electromagnetism.

and if V is the measure of the volume of a rectangular solid (parallelepiped) whose edges have the measures a, b, c, then

$$V = abc. \qquad (2.3,4)$$

The consistent unit of velocity is such that unit distance is moved in unit time. Suppose, then, that a distance whose measure is l is described in a time interval whose measure is t. Then, if the motion is uniform, the distance whose measure is l/t will be described in unit time and the measure of the velocity is therefore l/t. A body whose mass is unity and whose velocity is unity has unit momentum. Then if the measure of the mass of a body is m and the body moves uniformly through a distance whose measure is l in a time whose measure is t the measure of its momentum is ml/t.

The unit of force is such that it imparts unit momentum to a body in unit time. Then a force whose measure is f when acting upon a body whose mass is of measure m imparts to it momentum whose measure is f or velocity whose measure is f/m in the unit of time. In other words, this force imparts an acceleration to the body whose measure is f/m.

In the manner sketched above we can establish the definitions of the consistent units of all the quantities which arise in mechanics, based on the units of length, time and mass, which are treated as fundamental. There is, however, no necessity to use this particular set of fundamental units. We could, for instance, take the units of force, length and time as fundamental and define the unit of mass as such that unit force imparts unit acceleration to it.

The great advantage of adopting a consistent set of units is that the equations representing symbolically the relations between the measures of the quantities which appear in any physical problem hold good quite irrespective of the choice of the fundamental units. Equations related to physical problems and which possess this property have been called *complete* by Buckingham. Incomplete equations are sometimes used by engineers but, although the equations may be valid in a particular field of application, they are not applicable universally. As an example, in dealing with hydraulic forces we may omit the density of the fluid from the formulae on the ground that the density of water is, with sufficient accuracy, a mere constant. The resulting equations will be incomplete; they are valid for water but are not valid for any fluid whose density differs from that of water.

There is no compulsion to use a system of consistent units, and anyone making physical calculations has complete liberty to adopt whatever units he pleases. However, the chances of error are certainly greatly increased when inconsistent units are used, and the communication of results to others in a correct and unambiguous manner is rendered more difficult.

2.4. The Measure Formula of a Unit and the Physical Dimensions of a Quantity. Let us take a new unit of length equal to L of the original units of length, i.e. L is its measure in terms of the original unit. The new consistent unit of area is a square whose side contains L of the original units of length and it therefore contains L^2 of the original consistent units of area. Thus the new unit of area has the measure L^2 in terms of the original unit. We shall express this by saying that the consistent unit of area has the *measure formula* L^2.

Again, let us take a new unit of time containing T of the original units of time. Then the new consistent unit of velocity is such that L of the original units of length are described in T of the original units of time. Therefore the new consistent unit of velocity has the measure L/T in terms of the original consistent unit, so the measure formula of the consistent unit of velocity is LT^{-1}. Similarly, if the new unit of mass has the measure M in terms of the original unit of mass, the new consistent unit of momentum will have the measure formula MLT^{-1}.

The foregoing examples lead us to the *definition* : the *measure formula* of a unit is the mathematical expression for the measure of the new consistent unit in terms of the original consistent unit, the variables being the measures of the new fundamental units in terms of the original fundamental units. We shall also say that any physical quantity has the same measure formula as the consistent unit in terms of which it can be measured. By convention, the measure formula of a quantity, so defined, is also called the expression for the *physical dimensions* of the quantity.

Since the choice of the fundamental units is to some extent arbitrary, as pointed out in § 2.3, the physical dimensions of a quantity are to that extent arbitrary.

Table 2.4,1 gives the measure formulae or physical dimensions of the quantities commonly appearing in mechanics, where the fundamental units are chosen to be those of mass, length and time. In the formulae M, L and T stand for the measures of the new units of mass, length and time, respectively, in terms of the original units.

It is convenient to have a special symbol to denote dimensional equality and we shall adopt ⌓, that is the letter D turned on its side* ; it is easy to remember that ⫐ stands for dimensional equality. Thus we shall have, for example,

$$\text{Force} \;⩦\; MLT^{-2}$$
$$\text{Work} \;⩦\; ML^2T^{-2}$$
$$\text{Moment of inertia} \;⩦\; \text{Mass} \times \text{Area}.$$

It may be well to emphasize that ⩦ does not indicate either numerical equality or physical identity, but merely identity as regards measure formulae. Thus if we have

$$M^aL^bT^c \;⩦\; M^\alpha L^\beta T^\gamma$$

then $a = \alpha$

$b = \beta$

and $c = \gamma$.

The symbol ⩦ will be consistently used throughout this book.

We have pointed out in § 2.2 that the measure of any given physical quantity is inversely proportional to the magnitude of the unit in terms of which it is measured. Suppose then that new fundamental units are taken whose measures are M, L and T in terms of the original units of mass, length and time, respectively. The magnitude of the new consistent derived unit is equal to its measure formula with these values substituted and the new measure of the given quantity is therefore equal to the original measure *divided by* the measure formula. We thus have the rule:

New measure = original measure divided by the measure formula of the appropriate unit, with the numerical values of M, L and T substituted, where these are the measures of the new fundamental units in terms of the old.

If the fundamental units adopted are other than mass, length and time the same rule holds, *mutatis mutandis*.

The foregoing rule expresses the fundamental use of a measure formula and we shall illustrate this by a few examples.

EXAMPLE 1. Express the *poundal* (the consistent unit of force when the units of mass, length and time are the pound, foot and second, respectively) in terms of the *dyne* (the consistent unit of force when the units of mass, length and time are the gram, centimetre and second, respectively).

The data are M = number of grams in one pound
$= 453{\cdot}6$
L = number of centimetres in one foot
$= 30{\cdot}48$
$T = 1$ (unit of time unchanged).

* This symbol was used by the author in "A Review of Dimensional Analysis", *Engineering*, pp. 533 and 556, June, 1949.