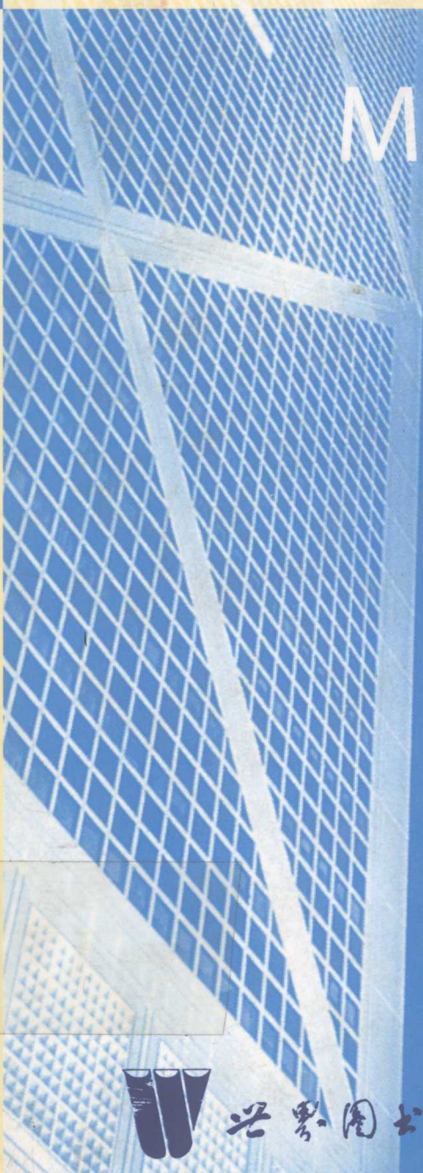
Springer Finance

Yue-Kuen Kwok

Mathematical Models of Financial Derivatives



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with 40 Figures and 2 Tables

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Preface

Objectives and Audience

In the past two decades, we have witnessed the revolutionary period in the trading of financial derivative securities in financial markets and the phenomenal surge in research activities in derivative pricing theory. Leading edge banking and financial firms around the globe are hiring science experts who can use advance analytical and numerical techniques to price financial derivatives and manage portfolio risks, a phenomenon coined as "Rocket Sciences on Wall Street". Such developments have spurred the growth of new degree programs in mathematical and computational finance in both North America, Europe and the Far East. This book is thus written to meet the demands of these students.

In this book, the derivative products are modelled as partial differential equation models from the perspectives of an applied mathematician. Analytical solutions to these derivative pricing models are sought, together with solution by numerical techniques when appropriate. This book includes quite a comprehensive coverage of derivative products commonly traded in financial markets, and the latest development of option pricing methodologies and algorithms. Fundamental concepts in financial economics which are necessary to the understanding of the derivative pricing theory are included, but possibly in a less formal style compared to other similar books on mathematical finance. Advanced concepts in probability theory which are essential for more in-depth understanding of derivative pricing theory, like the martingale theory and stochastic differential calculus, are briefly introduced in the book. The level of mathematics in this book is tailored to readers with preparation at the advanced undergraduate level of science and engineering majors. Other audience of this book include practitioners in financial engineering who seek the latest development in option pricing techniques, and scientists and mathematicians who would like to start exploring into the theory of derivative pricing.

The readers are assumed to have some basic knowledge of linear partial differential equations, probability and statistics, and numerical methods. No prior knowledge in finance theory and options is assumed.

Guide to the Chapters

This book contains seven chapters, each divided into a number of sections. Every chapter is ended with a set of well thought-out exercises. On one hand, the exercises provide the stimulation for refreshing concepts and knowledge acquired from the text; and on the other hand, they help lead the readers to results and concepts found scattered in recent journal articles in the field of derivative pricing.

The first chapter introduces general characteristics of financial derivatives. Fundamental concepts in financial option theory, like arbitrage, hedging, self-financing strategy, are discussed and definitions of these terms are presented. It is then followed by the discussion of the assumptions of asset price dynamics and the mathematical formulation of stochastic processes, in particular, the Geometric Brownian process. The Ito lemma, a basic tool used for the evaluation of stochastic differentials, is derived. The riskless hedging principle and risk neutrality are the cornerstones of the Black-Scholes-Merton pricing theory. The precise meaning and implication of these concepts are examined in the last section. The chapter is ended with the derivation of the renowned Black-Scholes equation.

In Chapter 2, the Black-Scholes equation is solved for different European-style derivative securities, which are distinguished by their contractual specifications. Under certain idealized assumptions, closed form solutions for the Black-Scholes equation can be found. The greeks of these Black-Scholes formulas are derived and their financial interpretations are discussed. The extension of the Black-Scholes model, incorporating the effects of discrete dividends, transaction costs, time-dependent interest rate and volatility are considered. Some of the issues of pricing biases of the Black-Scholes model are addressed. The last section deals with the discussion of pricing models for options on futures contracts.

The pricing of options that are multivariate in nature are considered in Chapter 3. Examples of such multi-state options include the index options, basket options, cross currency options, exchange options and options on the extremum of several risky assets. The general Black-Scholes equation for options with multiple underlying assets is developed. Analytic formulas for a variety of multi-state options are derived.

Chapter 4 is concerned with the pricing of the American-style options. The characterizations of the optimal exercise boundary associated with the American option models are presented. Special attentions are paid to examine the behaviors of the exercise boundary right before and after a discrete dividend payment, and immediately prior to expiry. The optimality conditions for the determination of the early exercise boundary are discussed. The early exercise premium is shown to be expressible in terms of the exercise boundary in the form of an integral. Several analytical approximation methods are discussed for the valuation of American options.

Examples of option models which lend themselves to closed form solutions are limited; and frequently, option valuation must be resorted to numerical procedures. The common numerical methods employed in option valuation are the binomial schemes, finite difference algorithm and Monte Carlo simulation. These numerical methods are discussed in Chapter 5. The primary essence of the binomial models is the simulation of the continuous asset price movement by a discrete random walk model. The finite difference approach seeks the discretization of the differential operators in the Black-Scholes equation. The Monte Carlo method simulates the random movement of the asset prices. It provides a probabilistic solution to the option pricing problems. An account of option pricing algorithms using these approaches is presented.

Path dependent options are option contracts where their payouts are related to movements in the price of underlying asset during the life of the option. The common examples are the barrier options, Asian options and lookback options. Chapter 6 presents the mathematical approaches for their valuation, including analytic approximation techniques and numerical procedures when analytic formulas do not exist.

Chapter 7 deals with the pricing of bonds and interest rate derivatives. Various classes of interest rate models are introduced, starting with the Vasicek mean reversion model, Cox-Ingersoll-Ross model, and extending to other multi-factor models. The discussion is extended to no-arbitrage models where the initial term structures are taken as inputs into the models. Pricing models of swaps, caps, floors, commodity-linked bonds and convertible bonds are also considered.

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I would like to thank Lixin Wu and Hong Yu for many hours of discussions about different aspects of the topics covered in the book. Also, I thank the students who read the draft version of the book as lecture notes. Their comments on the style and content further improve the book. Finally, my sincere thanks go to Ms Odissa Wong who has helped me in the typesetting of the manuscript, and entertained seemingly endless corrections.

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CHAPTER 1
General Characteristics of Financial Derivative Models

In the past two decades, we have witnessed the revolutionary period in the trading of *financial derivative securities* or *contingent claims* in financial markets around the world. A derivative security may be defined as a security whose value depends on the values of other more basic underlying variables, which may be the prices of traded securities, prices of commodities or stock indices. The two most common derivative securities are futures and options. A *forward contract* (called a *futures contract* if traded on exchanges) is an agreement between two parties to buy or sell an asset at a certain time in the future for a predetermined price while an *option* gives the holder the *right* (but not the obligation) to buy or sell an asset by a certain date for a predetermined price. There has been a great proliferation in the number and variety of derivative securities and new derivative products are being invented continuously. The construction of the theoretical framework for the pricing of new derivative securities has been one of the major challenges in the field of mathematical finance. The theoretical studies of financial derivative securities and their risk management have been commonly known as *Rocket Science* on Wall Street.

In this book, we concentrate on the study of pricing models for financial options. Option trading forms an integrated part in portfolio management in financial firms. Indeed, many financial strategies and decisions can be analyzed from the perspective of options. We explore the characteristics of various types of options and discuss the theoretical framework within which the fair prices of options can be determined.

In Sec. 1.1, we introduce some basic types of options traded in financial markets and present definitions of terms commonly used in the financial option theory, such as self-financing strategy, arbitrage, hedging, etc. Also, we discuss various trading strategies associated with the use of options and their combinations. In Sec. 1.2, we deduce rational boundaries on option values without any assumptions on the stochastic behavior of the prices of the underlying assets. The effects of the early exercise feature and dividend payments on option values are also discussed. It is then followed by the discussion of the assumptions of asset price dynamics and mathematical formulation of stochastic processes, in particular, the *Geometric Brownian process*. The *Ito lemma*, the basic tool used for the evaluation of stochastic differentials, is derived. A brief discussion on martingale, a type of stochastic process used to describe "fair" games, is also included. In the last section, we introduce the concepts of *riskless hedging* and *risk neutrality*, and present the derivation of the renowned *Black-Scholes equation* in option pricing theory. One can argue that the fair or arbitrage option price can be interpreted as the discounted expectation of its terminal payoff under an adjusted probability distribution. The adjusted probability distribution is precisely the distribution under which the asset price follows a martingale. Finally, we end the chapter by showing that the position of an option can be replicated by a self-financing dynamic trading strategy with assets and bonds only. The fair price of an option is simply the value of its replicating portfolio.

1.1 Financial options and their trading strategies

The initial discussion of option pricing theory focuses on revealing the definition and meaning of different terms in option trading. Options are classified either as a call option or a put option. A *call* (or *put*) option is a contract which gives the holder the *right* to buy (or sell) a prescribed asset, known as the *underlying asset*, by a certain date (*expiration date*) for a predetermined price (commonly called the *exercise price* or the *strike price*). Since the holder is given the right, but not the obligation, to buy or sell the asset, he will make the decision depending on whether the deal is favorable to him or not. The option is said to be *exercised* when the holder chooses to buy or sell the asset. If the option can only be exercised on the expiration date, then the option is called a *European* option, while if the exercise is allowed at any time prior to the expiration date, then it is called an *American* option (these terms have nothing to do with their continental origins). The simple call and put options with no special features are commonly called *plain vanilla options*. Also, we have options coined with names like *Asian option, lookback option, barrier option* etc. Their precise definitions will be given in subsequent chapters.

The other party to the holder of the option in the contract is called the *writer* of the option. The holder and the writer are said to be in the *long* and *short* positions of the option contract, respectively. Unlike the holder, the writer does have an obligation with regard to the option contract, say, the writer must sell the asset if the holder chooses in his favor to buy the asset. This is a zero-sum game. The holder gains from the loss of the writer or vice versa.

Terminal payoffs

Consider a European call option with strike price X and let S_T denote the price of the underlying asset on the date of expiration T. If $S_T > X$, then the holder of the call option will choose to exercise since he can buy the asset, which is worth S_T dollars, at the cost of X dollars. The gain to the holder from the call option is then $S_T - X$. However, if $S_T \leq X$, then the holder will forfeit the right to exercise the option since he can buy the asset in the market at a cost less than or equal to the predetermined strike price X. The terminal payoff from the long position (holder's position) in a European call is then

$$\max(S_T - X, 0).$$

Similarly, the terminal payoff from the long position in a European put can be shown to be

$$\max(X - S_T, 0),$$

since the put will be exercised only if $S_T < X$, under which the asset is sold at a higher price of X instead of S_T. In both call and put options, the terminal payoffs are non-negative. These properties reflect the very nature of options whereby they are exercised only if this results in positive payoffs.

Option premium

Since the writer of an option has the potential liabilities in the future, he must be compensated by an up-front premium payment of the option by the holder when they both enter into the option contract. An alternative viewpoint is that since the holder is guaranteed to receive a non-negative terminal payoff, he must pay a premium in order to enter into the game of the option. The natural question is: What should be the fair option premium (usually called option price or option value) so that the game is fair to both parties of writer and holder? Another but deeper question: What should be the optimal strategy to exercise prior the expiration date for an American option?

Throughout the remainder of this book, we shall discuss the construction of option pricing models for different types of financial options. At least, the option price is easily seen to depend on the strike price, the time to expiry and the current asset price. The less obvious factors for the pricing models are the prevailing *interest rate* and the degree of randomness of the asset price, commonly called the *volatility*.

Self-financing strategy

Suppose an investor holds a portfolio of securities, such as a combination of options, stocks and bonds. As time passes, the value of the portfolio changes since the prices of the securities change. Besides, the trading strategy of the investor affects the portfolio value, for example, by changing the proportions of the securities in the portfolio and adding or withdrawing funds from the portfolio. An investment strategy is said to be *self-financing* if no extra funds are added or withdrawn from the initial investment. The cost of acquiring more units of one security in the portfolio is completely financed by the sale of some units of another security within the same portfolio.

No-arbitrage principle
One of the fundamental concepts in the theory of option pricing is the absence of arbitrage opportunities, which is called the *no-arbitrage principle*. As an illustrative example of an arbitrage opportunity, suppose the prices of a given stock in Exchanges A and B are listed at \$99 and \$101, respectively. Assuming there is no transaction cost, one can lock in a riskless profit of \$2 per share by buying at \$99 in Exchange A and selling at \$101 in Exchange B. The trader who engages in such transactions is called an *arbitrageur*. If the financial market functions properly, such an arbitrage opportunity cannot occur since the traders are well alert and they immediately respond to compete away such an opportunity. However, when there is transaction cost, which is a common form of market friction, the small difference in prices may persist. For example, if the transaction costs for buying and selling per share in Exchanges A and B are both \$1.50, then the total transaction costs of \$3 per share will discourage arbitrageurs to seek the arbitrage opportunity arising from the price difference of \$2.

Stated in more rigorous language, an *arbitrage opportunity* can be defined as a self-financing trading strategy requiring no initial investment, having no probability of negative value at expiration, and yet having a possibility of a positive payoff.

Volatile nature of options
Option prices are known to respond in an exaggerated scale to changes in the underlying asset price. To illustrate the claim, we consider a call option which is near the time of expiration and the strike price is \$100. Suppose the current asset price is \$98, then the call price is close to zero since it is quite unlikely for the asset price to increase beyond \$100 within a short period of time. However, when the asset price is \$102, then the call price near expiry is about \$2. Hence, the option price is seen to be more volatile than the underlying asset price. In other words, the trading of options leads to more price action per each dollar of investment than the trading of the underlying asset. A precise analysis of the volatile nature of a particular option requires the detailed knowledge of the relevant valuation model for the option (see Eqs. (18a,b) in Sec. 2.1).

Hedging
If the writer of a call does not simultaneously own any amount of the underlying asset, then he is said to be in a *naked position* since he may be hard hit with no protection when the asset price rises sharply. However, if the call writer owns some amount of the underlying asset, the loss in the short position of the call when asset price rises can be compensated by the gain in the long position of the underlying asset. This strategy is called *hedging* where the risk in a portfolio is monitored by taking opposite directions in two assets which are highly negatively correlated. In a *perfect hedge* situation, the *hedger* combines a risky option and a risky underlying asset in appropriate

proportions to form a riskless position that performs like a default-free bond which earns the riskless interest rate. The *riskless hedging principle* forms the cornerstone of option pricing theory (see Sec. 1.4).

1.1.1 Trading strategies involving options

We have seen in the above simple hedging example how the combined use of an option and the underlying asset can monitor risk exposure. Now, we would like to examine in more detail various strategies of portfolio management with options and the underlying asset as the basic financial instruments in a portfolio. Here, we confine our discussion of portfolio strategies to the use of European vanilla call and put options. The understanding of the portfolio strategies for more sophisticated types of options requires a detailed analysis of the relevant valuation models.

The simplest approach to analyze a portfolio strategy is the construction of the corresponding *terminal profit diagram*, which shows the profit from holding the options and the underlying asset until the date of expiration as a function of the underlying asset price at expiry. This simplified analysis is applicable only to a portfolio which contains options with the same date of expiration and on the same underlying asset.

Covered calls and protective puts

In the above hedging example, we construct a portfolio which consists of a short position (writer) in one call option plus a long holding of one unit of the underlying asset. This investment strategy is known as *writing a covered call.* Let c denote the premium received by the writer when selling the call and S_0 the asset price at the start of the option contract ($S_0 > c$, see Eq. (13), Sec. 1.2). The initial portfolio value is then $S_0 - c$. Recall that the terminal payoff for the call is $\max(S_T - X, 0)$, where S_T is the asset price at expiry and X is the strike price. The portfolio value at expiry is $S_T - \max(S_T - X, 0)$ and so the profit at expiry is given by

$$
\begin{aligned}
&S_T - \max(S_T - X, 0) - (S_0 - c) \\
&= \begin{cases} (c - S_0) + X & \text{when } S_T \geq X \\ (c - S_0) + S_T & \text{when } S_T < X. \end{cases}
\end{aligned} \tag{1}
$$

Observe that when $S_T \geq X$, the profit remains at the constant value $(c - S_0) + X$, and when $S_T < X$, the profit grows linearly with S_T. The corresponding profit diagram for a covered call at expiry is illustrated in Fig. 1.1.

The *reverse of a covered call* is a portfolio which consists of a long position (holder) in one call option and a short position in one unit of the asset. The short position in the asset means that the portfolio owes the asset and thus refers to the selling of the asset that is not owned. The trading practice of borrowing a share, selling it, buying the share later and returning it to the owner is called *short selling*. The short sellers hope to profit from a price decline by selling before the decline and buying after the price falls. Usually, there are rules in stock exchanges that restrict the timing of the short selling and the use of the short sale proceeds. The profit at expiry for the reverse of a covered call is exactly negative to that of a covered call.

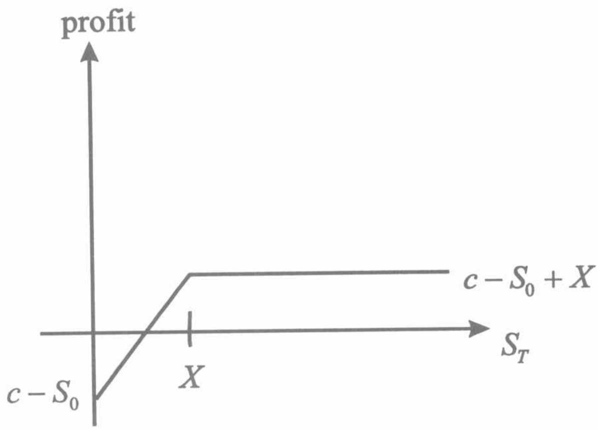

Fig. 1.1 Profit diagram of a covered call at expiry.

The investment portfolio that involves long positions in a put and the underlying asset is called a *protective put*. Let p denote the premium paid by the holder of the put to purchase the option. It can be shown similarly that the profit at expiry is given by

$$\begin{aligned} & S_T + \max(X - S_T, 0) - (p + S_0) \\ = & \begin{cases} -(p+S_0) + S_T & \text{when } S_T \geq X \\ -(p+S_0) + X & \text{when } S_T < X, \end{cases} \end{aligned} \tag{2}$$

and the corresponding profit diagram at expiry is shown in Fig. 1.2.

Is it meaningful to create a portfolio that involves long holding of a put and short selling of the asset? Such portfolio strategy will have no hedging effect since both positions in option and asset are in the same direction in risk exposure – both lose when the asset price increases.

Spreads

A spread strategy refers to a portfolio which consists of options of the same type (that is, two or more calls, or two or more puts) where some options are in the long position and others are in the short position. The two most basic strategies are the vertical spread (also called money or price spread) and the horizontal spread (also called time or calendar spread). In a *vertical spread*, one option is bought while another is sold, both on the same underlying asset and the same date of expiration but with different strike prices. A *horizontal*