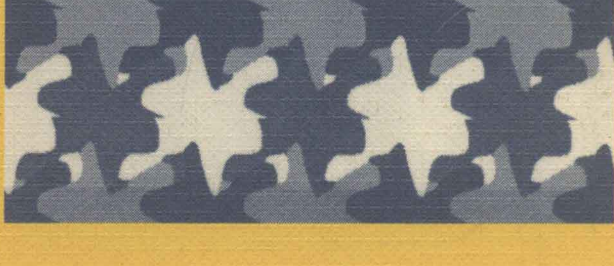
Lecture Notes in Mathematics

1576

Kazuaki Kitahara

# Spaces of Approximating Functions with Haar-like Conditions



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# Spaces of Approximating Functions with Haar-like Conditions

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*To Junko and Toshiya*

# Preface

Let $E$ be a function space with a norm $\|\cdot\|$ and let $G$ be a finite dimensional subspace of $E$. Then it is one of the principal themes in approximation theory to study the following problems: For each $f \in E$,

find $\tilde{f} \in G$ such that $E_G(f) = \|f - \tilde{f}\| = \inf_{g \in G} \|f - g\|$

and

estimate $E_G(f)$.

$G$ is called an approximating space and $\tilde{f}$ is said to be a best approximation to $f$ from $G$. If $G$ is chosen in the manner so that $E_G(f)$ is as small as possible and so that functions in $G$ are easy to handle, then $G$ is a good approximating space.

For example, in $C[a,b]$ (=the space of all real-valued continuous functions on $[a,b]$) with the supremum norm, spaces of polynomials with degree at most $n$ and spaces of continuous and piecewise linear functions with fixed knots are suitable for good approximating spaces. Čebyšev (or Haar) spaces and weak Čebyšev spaces are generalizations of these two spaces and play a central part when considering the above problems. In fact, properties, characterizations and generalizations of Čebyšev and weak Čebyšev spaces have been deeply studied during this century. Now, the theory of these spaces has matured.

In this book, as approximating spaces, we shall introduce Haar-like spaces, which are Haar and weak Čebyšev spaces under special conditions. And we shall study topics of subclasses of Haar-like spaces rather than general properties of Haar-like spaces, that is, classes of Čebyšev or weak Čebyšev spaces, spaces of vector-valued monotone increasing or convex functions and spaces of step functions.

Contents are mostly new results and rewritings of the following papers, 13, 14, 15, 16, 17 (Chapter 2), 7, 8, 9(Chapter 3), 17, 18 (Chapter 4), 4, 5 (Chapter 5), 2 (Appendix 1), where each number is its reference number. In Chapter 1, Haar-like spaces are defined and several examples of Haar-like spaces are given. In Chapter 2 and 3, for Čebyšev and Čebyšev-like spaces, we are concerned with characterizations, derivative spaces, separated representations, adjoined functions and best $L^1$-approximations. In Chapter 4, in a space of vector-valued functions of bounded variation, we consider best approximations by monotone increasing or convex functions. In Chapter 5, approximation by step functions is studied. In connection with Chapter 5, Dirichlet tilings and a certain property of the finite decomposition of a set are stated in Appendix 1 and 2, respectively. The readers can see further summary in the Introduction to each chapter.

I would like to express my heartfelt gratitude to emeritus Prof. Kiyoshi Iseki at Naruto Education University and Prof. Shirô Ogawa at Kwansei Gakuin University who taught me topological vector spaces and approximation theory and who have given me constant encouragement with high degree. I am indebted to Katsumi Tanaka, Takakazu Yamamoto and Hiroaki Katsutani for their help in preparing this manuscript, to Donna L. DeWick for checking the style of this manuscript and to the editors and staff of Springer-Verlag for their able cooperation.

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Saga, Japan
January, 1994

# Contents

# Chapter 1 Preliminaries

## 1.1 Introduction

Before stating the purpose of this chapter, we shall introduce a concrete problem in approximation theory. Let $C[a,b]$ be the space of all real-valued functions on a compact real interval $[a,b]$. $C[a,b]$ is endowed with the supremum norm $\|\cdot\|$,i.e., $\|f\| = \sup_{x\in[a,b]}|f(x)|$ for all $f \in C[a,b]$. For a finite dimensional subspace $G$ of $C[a,b]$, we consider the following problem: For a given $f \in C[a,b]$,

find $\tilde{f} \in G$ such that $\|f-\tilde{f}\| = \inf_{g\in G}\|f-g\|$,

in other words,

find the best approximations $\tilde{f}$ to $f$ from $G$.

$G$ is a space of approximating functions. It is well known that spaces spanned by the following systems $\{u_i\}_{i=1}^n$ in $C[a,b]$ are of good use to treat this problem. One is a generalization of systems of spline functions and the other is a generalization of systems of polynomials.

(1) $\{u_i\}_{i=1}^n$ is a system such that, for any $n$ distinct points $(a \le)x_1 < \ldots < x_n(\le b)$,

$$\det\begin{pmatrix} \sigma u_1(x_1) & \cdots & u_n(x_1) \\ \vdots & \cdots & \vdots \\ \sigma u_1(x_n) & \cdots & u_n(x_n) \end{pmatrix} \ge 0,$$

where $\sigma = 1$ or $-1$.

(2) $\{u_i\}_{i=1}^n$ is a system such that, for any $n$ distinct points $(a \le)x_1 < \ldots < x_n(\le b)$,

$$\det\begin{pmatrix} u_1(x_1) & \cdots & u_n(x_1) \\ \vdots & \cdots & \vdots \\ u_1(x_n) & \cdots & u_n(x_n) \end{pmatrix} \ne 0.$$

Systems of (1) are called weak Tchebycheff, weak Čebyšev or WT-systems and those of (2) are called Haar, Tchebycheff, Čebyšev or T-systems.(Unified terms are not used for these systems.) One useful property of $WT$ and $T$-systems is its marked characterization of best approximations from these spaces. When spaces spanned by $WT$ and $T$-systems are approximating spaces, best approximations in the above problem are chracterized as follows.

**Theorem 1.1.1.** *Let $\{u_i\}_{i=1}^n$ be a $WT$-system in $C[a,b]$ and let $G$ be the space spanned by $\{u_i\}_{i=1}^n$. For a given $f \in C[a,b]$, if there exist an $\tilde{f} \in G$ and $n+1$ points*

*$(a \leq)$ $x_1 < \cdots < x_{n+1}(\leq b)$ such that $\|f - \tilde{f}\| = \sigma(-1)^i(f(x_i) - \tilde{f}(x_i))$ ($\sigma = 1$ or $-1$), $i = 1, \ldots, n+1$, then $\tilde{f}$ is a best approximation to $f$ from $G$. Furthermore, each $f \in C[a,b]$ has a best approximation from $G$ satisfying this condition.*

**Theorem 1.1.2.** *Let $\{u_i\}_{i=1}^n$ be a $T$-system in $C[a,b]$ and let $G$ be the space spanned by $\{u_i\}_{i=1}^n$. Each $f \in C[a,b]$ has a unique best approximation from $G$ satisfying the condition stated in* Theorem 1.1.1.

These systems play an important role not only in characterizations of best approximations, but also in interpolation methods, moment spaces, totally positive kernels and so on.

Some generalizations of $WT$ and $T$-systems have already been defined and investigated. (e.g. Ault, Deutsch, Morris and Olson[1], Garkavi[4] e.t.c.) In this chapter, we will define other types of generalized $WT$ and $T$-systems. Furthermore, we mainly study these systems in the following chapters.

## 1.2 Definitions

We begin by giving definitions of Haar-like conditions.

**Definition 1.** Let $E$ be a real vector space and let $E^*$ be the algebraic dual space of $E$,i.e., the space of all real-valued linear functionals on $E$. For a given positive integer $n$, let $\{a_i\}_{i=1}^n$ be $n$ linearly independent elements in $E$ and let $\mathcal{F}$ be a subset of $(E^*)^n$ $(= \overbrace{E^* \times \cdots \times E^*}^{n})$.

(1) If, for any element $(x_1, \ldots, x_n) \in \mathcal{F}$, the $n$-th order determinant

$$D\begin{pmatrix} a_1 & \ldots & a_n \\ x_1 & \ldots & x_n \end{pmatrix} := \det\begin{pmatrix} x_1(a_1) & \ldots & x_1(a_n) \\ \vdots & \ldots & \vdots \\ x_n(a_1) & \ldots & x_n(a_n) \end{pmatrix} \neq 0,$$

then it is said that $\{a_i\}_{i=1}^n$ satisfies $H$ *condition with* $\mathcal{F}$ or is an $H$*-system with* $\mathcal{F}$(abbreviated $H_{\mathcal{F}}$*-system*). The space $[a_1, \ldots, a_n]$ spanned by elements of an $H_{\mathcal{F}}$-system $\{a_i\}_{i=1}^n$ is called an $H_{\mathcal{F}}$*-space*.

(2) If, for any element $(x_1, \ldots, x_n) \in \mathcal{F}$,

$$D\begin{pmatrix} \sigma a_1 & \ldots & a_n \\ x_1 & \ldots & x_n \end{pmatrix} > 0, \quad (\text{resp.} \geq 0)$$

where $\sigma = 1$ or $-1$, then it is said that $\{a_i\}_{i=1}^n$ satisfies *T condition* (resp. *WT condition*) *with* $\mathcal{F}$ or is a *$T_{\mathcal{F}}$-system* (resp. *$WT_{\mathcal{F}}$-system*). And, we call the space spanned by a $T_{\mathcal{F}}$-system (resp. $WT_{\mathcal{F}}$-system) a *$T_{\mathcal{F}}$-space* (resp. *$WT_{\mathcal{F}}$-space*).

(3) Let $\mathcal{F}_k = \{(x_1, \ldots, x_k) \mid (x_1, \ldots, x_n) \in \mathcal{F}\}$ for each $k$, $1 \le k \le n$. If $\{a_i\}_{i=1}^n$ is a system such that, for each $k$, $1 \le k \le n$, $\{a_i\}_{i=1}^k$ is an $H_{\mathcal{F}_k}$-system (resp. $T_{\mathcal{F}_k}$, $WT_{\mathcal{F}_k}$-system), then it is called a *complete $H_{\mathcal{F}}$-system* (resp. *complete $T_{\mathcal{F}}$, complete $WT_{\mathcal{F}}$-system*).

(4) For our convenience, when we take these three conditions in (1), (2) and (3) together, we call them *Haar-like conditions*. Analogously, we use the terms *Haar-like systems* and *Haar-like spaces*.

Now we add two approximating spaces in a normed space.

**Definition 2.** Let $E$ be a normed space with a norm $\|\cdot\|$ and let $M$ be a subset of $E$.

(1) For a given $x \in E$, if there exists an $x_0 \in M$ such that $\|x - x_0\| = \inf_{y \in M} \|x - y\|$, then $x_0$ is called a *best approximation to $x$ from $M$* or simply a *best approximation to $x$*. The set of all best approximations to $x$ from $M$ is denoted by $P_M(x)$.

(2) For a subspace $G$ of $E$, let $U_G = \{x | x \in E, P_G(x) \text{ is singleton }\}$. If $U_G = E$, $G$ is called a *$C$-space*, and if $E - U_G$ is a set of the first category in $E$, then $G$ is said to be an *$AC$-space*.

**Remark 1.** (1) $\{a_i\}_{i=1}^n$ is a $T_{\mathcal{F}}$-system if and only if it is a $WT_{\mathcal{F}}$ and $H_{\mathcal{F}}$-system. But every $WT_{\mathcal{F}}$ or $H_{\mathcal{F}}$-system is not always a $T_{\mathcal{F}}$-system.

(2) As for completeness of Haar-like systems, we can define a more specialized system such as order complete or Descartes systems (see Nürnberger[8; p.15]). But these systems are not addressed in this book.

(3) Let $\{a_i\}_{i=1}^n$ be a system of $E$ and let $\mathcal{F}$ be a subset of $(E^*)^n$. For any $(x_1, \ldots, x_n) \in \mathcal{F}$ and each $k$, $1 \le k \le n$, $x_i^{(k)}$ is a linear functional on $E$ such that $x_i^{(k)}(a_j) = x_i(a_j)$ if $1 \le i, j \le k$, $x_i^{(k)}(a_j) = 1$ if $i = j > k$, $x_i^{(k)}(a_j) = 0$ otherwise. If, for each $\{a_i\}_{i=1}^n$, we put $\mathcal{F}' = \{(x_1^{(k)}, \ldots, x_n^{(k)}) \mid (x_1, \ldots, x_n) \in \mathcal{F}, 1 \le k \le n\}$, then $\{a_i\}_{i=1}^n$ is a complete $H_{\mathcal{F}}$-system if and only if it is an $H_{\mathcal{F}'}$-system.

(4) The definition of $AC$-spaces is introduced by Stečkin[12].

(5) Best approximations in general normed spaces are studied in detail in Singer[11].

From Definition 1, the following statement immediately follows.

**Proposition 1.2.1.** *Let $E$ be a real normed space and let $E'$ be the topological dual space of $E$,i.e., the space of all real-valued continuous linear functionals on $E$. Let $\mathcal{F}$ be a connected subset of $(E')^n$, where $E'$ is endowed with the weak topology $\sigma(E', E)$. Then if $\{a_i\}_{i=1}^n$ is an $H_{\mathcal{F}}$-system in $E$, it is a $T_{\mathcal{F}}$-system.*

### 1.3 Examples of Haar-like Spaces

We give some function spaces and examples of Haar-like subspaces, which are studied in the following chapters.

**1.** Let $E$ be a real normed space and let $E'$ be the topological dual space of $E$. $S_{E'}$ denotes the closed unit ball in $E'$ and the set of extreme points of $S_{E'}$ is denoted by $extS_{E'}$. An $n$-dimensional subspace $M$ of $E$ is called an *interpolating subspace* if, for any $n$ linearly independent functionals $x_1, \ldots, x_n$ in $extS_{E'}$ and any $n$ real scalars $c_1, \ldots, c_n$, there is a unique element $a \in M$ such that $x_i(a) = c_i$ for $i = 1, \ldots, n$. Ault, Deutsch, Morris and Olson[1] gave the definition of interpolating subspaces and studied best approximations from interpolating subspaces in detail.

As a subset $\mathcal{F}$ of $(E')^n$, setting $\mathcal{F} = \{(x_1, \ldots, x_n) |\ x_1, \ldots, x_n$ are linearly independent functionals in $extS_{E'}\}$ , we can consider that every interpolating subspace is an $H_{\mathcal{F}}$-space.

Interpolating spaces are closely related with $C$-spaces.

**Proposition 1.3.1.**(Ault, Deutsch, Morris and Olson[1; Theorem 2.2]) *Let $M$ be a finite dimensional subspace of a real normed space. If $M$ is an interpolating space, then it is a $C$-space.*

**2.** For a set $X$, $F(X)$ denotes the space of all real-valued functions on $X$. We easily see that each point $x$ in $X$ is a linear functional on $F(X)$ such that $x(f) = f(x)$ for all $f \in F(X)$. Let $\{A_i\}_{i=1}^n$ be an $n$-decomposition of $X$,i.e., $A_i \cap A_j = \emptyset$ for $i \neq j$ and $\cup_{i=1}^n A_i = X$, and let $\chi_{A_i}(x)$, $i = 1, \ldots n$, be the characteristic function of $A_i$. Then $[\chi_{A_1}, \ldots, \chi_{A_n}]$ is an $n$-dimensional subspace of $F(X)$ which consists of step functions on $X$.

If we consider a subset of $(E^*)^n$, $\mathcal{F} = \{(x_1, \ldots, x_n) | x_i \in A_i, i = 1, \ldots, n\}$, $\{\chi_{A_i}\}_{i=1}^n$ can be regarded as a $T_{\mathcal{F}}$-system.

Problems of approximation by step functions will be studied in Chapter 5.

**3.** Let $T$ be a partially ordered set with an order $\leq$. As a subset of $(F(T)^*)^n$, setting $\mathcal{P} = \{(x_1, \ldots, x_n) \mid x_1 < \cdots < x_n,\ x_i \in T,\ i = 1, \ldots, n\}$ ($x < y$ means that $x \leq y$ and $x \neq y$.), we can consider $H_{\mathcal{P}}$, $T_{\mathcal{P}}$ and $WT_{\mathcal{P}}$-systems. When $T$ is a linearly ordered set, Zielke[14] studied many properties of $H_{\mathcal{P}}$, $T_{\mathcal{P}}$ and $WT_{\mathcal{P}}$-systems in $F(T)$ (Zielke[14] uses other terminologies for these systems) which consist of discontinuous functions.

From Lemma 1.2 and Lemma 4.1 in Zielke[14], we have

**Proposition 1.3.2.** *Let $T$ be a linearly ordered set containing at least $n+1$ points and $\{u_i\}_{i=1}^n$ be a system in $F(T)$.*

(1) *$\{u_i\}_{i=1}^n$ is an $H_{\mathcal{P}}$-system if and only if any $u \in [u_1, \ldots, u_n] - \{0\}$ has at most $n-1$ zeros.*

(2) *$\{u_i\}_{i=1}^n$ is a $WT_{\mathcal{P}}$-system if and only if no $u \in [u_1, \ldots, u_n]$ has an alternation of length $n+1$,i.e., there do not exist $n+1$ points $x_1 < \cdots < x_{n+1}$ in $T$ such that $(-1)^i u(x_i)$ is positive for $i = 1, \ldots, n+1$ or negative for $i = 1, \ldots, n+1$.*

A necessary and sufficient condition of $T_{\mathcal{P}}$-systems immediately follows from the conditions in (1) and (2) in Proposition 1.3.2. Clearly, Tchebycheff systems (resp. weak Tchebycheff systems) in 1.1 are identical with $T_{\mathcal{P}}$-systems (resp. $WT_{\mathcal{P}}$-systems).

**4.** For a Hausdorff topological space $X$, $C(X)$ denotes the space of all real-valued continuous functions on $X$ and $C_0(X)$ denotes the subspace of $C(X)$ which consists of functions $f$ such that $\{x \in X \mid |f(x)| \geq \epsilon\}$ is compact for each $\epsilon > 0$.

In this book, we mainly treat $X$ as a subset of the real line $R$. In particular, when $X$ is a nondegenerate compact interval $[a, b]$, we use the notation $C[a, b]$ instead of $C([a, b])$. And as another function space on $[a, b]$, we denote the space of all real-valued Lebesgue integrable functions on $[a, b]$ by $L^1[a, b]$.

As is stated in 1.1, properties of $H_{\mathcal{P}}$, $T_{\mathcal{P}}$ and $WT_{\mathcal{P}}$-systems in $C[a, b]$ (Other terminologies are used for these systems) have been profoundly studied. We can observe many good properties of these systems in books and journals related to approximation theory.(e.g. Cheney[2], Davis[3], Karlin and Studden[5], Lorentz[7], Watson[13] etc.)

Now we introduce other types of systems in $C[a, b]$ or $L^1[a, b]$. Let $\mathcal{S}$ be the set of all nondegenerate subintervals of $[a, b]$. For $I, J \in \mathcal{S}$, if the interior points of $I$ equal those of $J$, we write $I = J$, and if $I \cap J$ has no interior points and $x \leq y$ for all $x \in I$

and $y \in J$, then it is denoted by $I < J$ for this relation. By this, $(\mathcal{S}, \leq)$ is a partially ordered set, and $\leq$ means $=$ or $<$. For each $I \in \mathcal{S}$, we define a linear functional $u_I$ on $C[a,b]$ or $L^1[a,b]$ such that $u_I(f) = \int_I f(x)dx$ for all $f \in C[a,b]$ or $L^1[a,b]$ and let $\mathcal{I} = \{ (u_{I_1}, \ldots, u_{I_n}) \mid I_i \in \mathcal{S},\ i = 1, \ldots, n,\ I_1 < \cdots < I_n \}$. Then we can consider $H_{\mathcal{I}}$, $T_{\mathcal{I}}$ and $WT_{\mathcal{I}}$-systems in $C[a,b]$ or $L^1[a,b]$.

The readers can easily obtain the following proposition from the above definition.

**Proposition 1.3.3.** *Let $\{u_i\}_{i=1}^n$ be a system in $C[a,b]$ or $L^1[a,b]$. $\{u_i\}_{i=1}^n$ is an $H_{\mathcal{I}}$-system if and only if it is a $T_{\mathcal{I}}$-system.*

Shi[10] proposes a variation of $L$ approximation which maintains almost all of the Chebyshev theory and considers best approximations by $H_{\mathcal{I}}$-systems ($= QT$-systems in Shi[10]). Basic properties of $H_{\mathcal{I}}$-systems in $C[a,b]$ are studied in Kitahara[6]. Further properties of $H_{\mathcal{I}}$ and $WT_{\mathcal{I}}$-systems will be investigated in Chapter 2 and 3.

**5.** Let $C^{n-1}[a,b]$ $(n \geq 1)$ be the subspace of $C[a,b]$ which consists of $n-1$ times continuously differentiable functions and let $\{u_i\}_{i=1}^n$ be a system in $C^{n-1}[a,b]$. For any $\mathbf{x} = (x_1, \ldots, x_n)$ with $a \leq x_1 \leq \ldots \leq x_n \leq b$, we define the following linear functionals $z_i^{\mathbf{x}}$ on $C^{n-1}[a,b]$:

$$z_i^{\mathbf{x}}(f) = f^{(k_i)}(x_i) \quad \text{for } f \in C^{n-1}[a,b],\ i = 1, \ldots, n,$$

where $k_i = \max \{ j \mid j$ is a nonnegative integer and $x_i = \cdots = x_{i-j}\}$, $i = 1, \ldots, n$. If $\{u_i\}_{i=1}^n$ satisfies the condition that

$$D\begin{pmatrix} \sigma u_1 & \ldots & u_n \\ z_1^{\mathbf{x}} & \ldots & z_n^{\mathbf{x}} \end{pmatrix} > 0$$

for all $x = (x_1, \ldots, x_n)$ with $a \leq x_1 \leq \ldots \leq x_n \leq b$, then $\{u_i\}_{i=1}^n$ is called an extended Tchebycheff system of order $n$ or simply an extended Tchebycheff system (see p.6 in Karlin and Studden[5] and p.4 in Nürnberger[8]). Setting $\widetilde{\mathcal{P}} = \{(z_1^{\mathbf{x}}, \ldots, z_n^{\mathbf{x}}) \mid a \leq x_1 \leq \ldots \leq x_n \leq b\}$, every extended Tchebycheff system is identical with a $T_{\widetilde{\mathcal{P}}}$-system. In particular, if $\mathbf{x} = (t, \ldots, t)$, $t \in [a,b]$, $D\begin{pmatrix} \sigma u_1 & \ldots & u_n \\ z_1^{\mathbf{x}} & \ldots & z_n^{\mathbf{x}} \end{pmatrix}$ denotes Wronskian determinants $W(u_1, \ldots, u_n)(t)$ of $\{u_i\}_{i=1}^n$ at $t \in [a,b]$.

**6.** We give an example of generalized convex functions in $F[a,b]$ by using $WT_{\mathcal{P}}$-condition. Let $\mathcal{U} = \{u_i\}_{i=1}^n$ be a system in $F[a,b]$ consisting of linearly independent functions. If $f$ is a function in $F[a,b]$ such that $\{u_i\}_{i=1}^n \cup \{f\}$ is a $WT_{\mathcal{P}}$-system, then

$f$ is called a $\mathcal{U}$-convex function. When $\mathcal{U} = \{1, x, x^2, \ldots, x^{n-1}\}$, $\mathcal{U}$-convex functions are called $n$-convex functions.(see Roberts and Varberg[9], Zwick[15])

Analogously, we can consider vector-valued $\mathcal{U}$-convex functions. Let $\mathcal{U} = \{u_i\}_{i=1}^n$ be a system in $F[a, b]$ consisting of linearly independent real-valued functions, and let $E$ be an ordered real vector space. If $f$ is an $E$-valued function on $[a, b]$ such that, for any $(x_1, \ldots, x_{n+1}) \in \mathcal{P}$, the $n+1$-th order determinant

$$\det \begin{pmatrix} u_1(x_1) & \cdots & u_1(x_{n+1}) \\ \vdots & \cdots & \vdots \\ u_n(x_1) & \cdots & u_n(x_{n+1}) \\ f(x_1) & \cdots & f(x_{n+1}) \end{pmatrix} \geq 0,$$

then we call $f$ an $E$-valued $\mathcal{U}$-convex function. In this book, we do not study properties of vector-valued $\mathcal{U}$-convex functions, but we will treat approximation by vector-valued 1-convex or 2-convex functions in Chapter 4.

**Remark 2.** (1) Let $\mathcal{U}$ be an extended complete Tchebycheff system. Karlin and Studden studied properties of the set (= cone) of all real-valued $\mathcal{U}$-convex functions in depth(see Karlin and Studden[5; Chapter XI]).

(2) Any system in a real vector space can be an $H_{\mathcal{F}}$-system or $WT_{\mathcal{F}}$-system for some subset $\mathcal{F} \subset (E^*)^n$. Hence, it is important to consider Haar-like systems under ideal subsets of $(E^*)^n$.

(3) We shall use the terminologies introduced here throughout this book.

### 1.4 Problem

**1.** Let $M$ be a finite dimensional subspace of a real normed space $(E, \|\cdot\|)$. If $M$ is an $H_{\mathcal{F}}$-space, where $\mathcal{F}$ is a subset of $(E')^n$ in example 1 in 1.3, then $M$ is a $C$-space. Similarly is there any subset $\mathcal{G}$ of $(E')^n$ such that an $H_{\mathcal{G}}$-space is an $AC$-space ?

# Chapter 2 Characterizations of Approximating Spaces of $C[a,b]$ or $C_0(Q)$

## 2.1 Introduction

Let $Q$ be a locally compact subset of $R$ and let $C_0(Q)$ and $C[a,b]$ be the function spaces defined in 1.3. $C_0(Q)$ and $C[a,b]$ are endowed with the supremum norm $\|\cdot\|$,i.e., $\|f\| = \sup_{x\in Q(x\in[a,b])} |f(x)|$ for each $f \in C_0(Q)(C[a,b])$. Let $\mathcal{P}$ and $\mathcal{I}$ be the subsets of $(C_0(Q)^*)^n$ or $(C[a,b]^*)^n$ defined in 1.3.

In this chapter, we introduce known characterizations of approximating spaces of $C_0(Q)$ or $C[a,b]$ and show other types of characterizations of these spaces.

In 2.2, basic properties of $H_{\mathcal{P}}$, $WT_{\mathcal{P}}$, $C$ and $AC$-spaces in $C_0(Q)$ are observed. In 2.3, we review characterizations of $T_{\mathcal{P}}(= H_{\mathcal{P}})$, $WT_{\mathcal{P}}$ and $H_{\mathcal{I}}(= T_{\mathcal{I}})$-spaces of $C[a,b]$. These are stated in terms of sets defined by best approximations. In general, all results in this section can not hold in $C_0(Q)$. In 2.4, material similar to 2.3 is introduced and some of the results in 2.3 are extended to $C_0(Q)$.

In 2.5 and 2.6, we consider different types of characterizations from those in 2.3 and 2.4. In 2.5, we show a characterization of $H_{\mathcal{P}}(= T_{\mathcal{P}})$-spaces of $C(R)$ in terms of appropriate decompositions of $R^2$. In 2.6, using the nonexistence theorem of best approximations from a closed subspace, we give a chracterization of a space spanned by an infinite complete $T_{\mathcal{P}}(= H_{\mathcal{P}})$-system. Finally, some problems related to these topics are stated in 2.7.

## 2.2 Approximating Spaces of $C_0(Q)$

Let $Q$ be a locally compact subset of $R$ and let $C_0(Q)$ be the Banach space defined in 2.1.

Let us recall that, for an $n$-dimensional subspace $G$ of $C_0(Q)$, if $U_G = C_0(Q)$, $G$ is said to be a $C$-space and if $C_0(Q) - U_G$ is a set of the first category in $C_0(Q)$, then $G$ is called an $AC$-space.

By Theorem 3.2 in Ault, Deutsch, Morris and Olson[1], we state

**Proposition 2.2.1.** *Let $G$ be an $n$-dimensional subspace in $C_0(Q)$. The following statements are equivalent:*

(1) *$G$ is an $H_{\mathcal{P}}$-space.*

(2) *For any $n$ distinct points $x_1, \ldots, x_n$ in $Q$ and any real numbers $c_1, \ldots, c_n$, there exists an $f \in G$ such that $f(x_i) = c_i$, $i = 1, \ldots, n$.*

(3) *$G$ is a $C$-space.*

(4) *$G$ is an interpolating subspace.*

The equivalence of (1) and (2) follows from a straightfoward application of the definition of $H_{\mathcal{P}}$-spaces. The equivalence of (1),(3) and (4) follows from Theorem 3.2 in Ault, Deutsch, Morris and Olson [1].

As a special case of Proposition 1.3.2, we state

**Proposition 2.2.2.** *Let $G$ be an $n$-dimensional subspace in $C_0(Q)$. The following statements are equivalent.*

(1) *$G$ is a $WT_{\mathcal{P}}$-space.*

(2) *No $f \in G$ has an alternation of length $n+1$.*

For every subset $A$ of $Q$, we associate the number $N_n(A)$ equal to the number of points of $A$, if this number does not exceed $n$, and equal to $n$ otherwise.

If $G$ is a finite dimensional $AC$-space, then we have

**Proposition 2.2.3.** *Let $Q$ be a locally compact subset of $R$ which contains at least $n$ points and let $G$ be an $n$-dimensional subspace of $C_0(Q)$. The following statements are equivalent:*

(1) *$G$ is an $AC$-space.*

(2) *On each open subset $O \subset Q$, at most $n - N_n(O)$ linearly independent functions in $G$ can vanish identically.*

(3) *There exists a dense subset $\mathcal{P}'$ of $\mathcal{P}$ such that $G$ is an $H_{\mathcal{P}'}$-space.*

**Proof.**

(1) $\leftrightarrow$ (2) Since $C_0(Q)$ is separable, the equivalence of (1) and (2) follows from the same proof of Theorem 1 in Garkavi[7].

(2) $\rightarrow$ (3) Suppose that $f_1, \cdots, f_n$ is a basis for $G$. It is sufficient to show a dense subset $\mathcal{P}'$ of $\mathcal{P}$ such that, for each $(x_1, \ldots, x_n) \in \mathcal{P}'$, $D\begin{pmatrix} f_1 & \cdots & f_n \\ x_1 & \cdots & x_n \end{pmatrix} \neq 0$.

Let $x_1, \ldots, x_n$ be any $n$ distinct points in $Q$ with $x_1 < \cdots < x_n$ and let $O_i$, $i = 1, \ldots, n$, be any open neighbourhood of $x_i$ such that $O_i \cap O_j = \emptyset$ for $i \neq j$. Without loss of generality, we may assume that each $O_i$, $i = 1, \ldots, n$, is a one point set or an