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Translated from the Russian
by George Yankovsky

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П. С. МОДЕНОВ

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CONTENTS

PREFACE

This text offers certain general methods of solving problems in elementary geometry and is designed for teachers of mathematics in secondary schools and also for senior students.

The present text includes material that goes beyond the scope of mathematics curricula for secondary schools (the use of complex numbers in plane geometry, inversion, pencils of circles and others).

The book consists of five chapters. The first four chapters deal with the application of vector algebra, analytic geometry, complex numbers and the inversion transformation to geometric problems. Chapter V contains a list of the basic definitions and formulas used in the first four chapters. Before starting a new chapter, the reader is advised to refresh his memory with the appropriate material of Chapter V. Some of the derivations of formulas given in Chapter V are familiar to senior students of secondary school. More detailed theoretical material can be found in the bibliography at the end of the book.

I wish here to remark on a supplement to vector algebra that was brought to my attention in 1930 by Professor Ya. S. Dubnov, my teacher at Moscow State University. It is that vector algebra in the plane has not been developed to the point that vector algebra in space has, and in order to remedy this situation in an oriented plane it is necessary to introduce the rotation of a vector through an angle of $+\pi/2$ (designated $[\mathbf{a}]$) and also a pseudoscalar (or cross) product $\mathbf{a} \times \mathbf{b}$ [or $(\mathbf{a}, \mathbf{b})$] of a vector $\mathbf{a}$ by a vector $\mathbf{b}$. Note that the linear vector function $A\mathbf{x}$ of a vector argument $\mathbf{x}$ possessing the property that $A\mathbf{x} \perp \mathbf{x}$ for any vector $\mathbf{x}$ has the form $A\mathbf{x} = \lambda[\mathbf{x}]$ (λ is an arbitrary number) in the plane, and $A\mathbf{x} = [\mathbf{a}, \mathbf{x}]$ ($\mathbf{a}$ is an arbitrary vector) in space. The cross product of vectors in the plane and in space may be defined as a polylinear scalar function (of two vectors in the plane and of three vectors in space) which is antisymmetric with respect to any pair of vectors — in the plane we have

$$A(\mathbf{x}, \mathbf{y}) = -A(\mathbf{y}, \mathbf{x});$$

in space we have

$$A(\mathbf{x}, \mathbf{y}, \mathbf{z}) = -A(\mathbf{y}, \mathbf{x}, \mathbf{z}), \quad A(\mathbf{x}, \mathbf{y}, \mathbf{z}) = -A(\mathbf{z}, \mathbf{y}, \mathbf{x}),$$
$$A(\mathbf{x}, \mathbf{y}, \mathbf{z}) = -A(\mathbf{x}, \mathbf{z}, \mathbf{y})$$

— and is normed (that is, it becomes $+1$ for some base).

This product may be defined as the result of two operations (in the plane and in space)

$$(\mathbf{a}, \mathbf{b}) = [\mathbf{a}] \cdot \mathbf{b}, \quad (\mathbf{a}, \mathbf{b}, \mathbf{c}) = [\mathbf{a}, \mathbf{b}] \cdot \mathbf{c}.$$

Although a free vector in geometry constitutes the *class* of all equivalent directed line segments, I will permit myself, in this book (in accordance with a very solid tradition) to identify a vector and a directed line segment as equal (as, for example, in arithmetic, where one regards as equal the fractions p/q and np/nq, where p, q, n are natural numbers). For this reason, in this text, two directed line segments that are collinear, have the same length, and are in the same direction will be termed equivalent or equal.

The idea of using complex numbers in plane geometry came to me in connection with some very interesting lectures on the theory of analytic functions delivered at Moscow University by Professor A. I. Markushevich, and also with a book on that subject by Markushevich. Also, since the 1940 s, papers have appeared regularly in mathematical journals in many countries illustrating how the use of complex numbers in plane geometry makes for rather simple solutions to complicated problems by relating the solutions to basic geometric transformations that are normally studied in secondary school (motion, the similarity transformation, circular transformations, including inversion).

A book by R. Deaux [2] appeared in France devoted specially to the problems taken up in Chapter III of this book. Since this methodology is not all represented in Soviet textbooks, I have given detailed explanations and calculations of the procedures. In this text I have made use of the work of R. Deaux, R. Blanchard, Gourmagschieg, V. Jebeau and others.

I believe that the contents of Chapter III is added proof of how much elementary mathematics loses if complex numbers are not brought into the picture. A consideration of the most elementary functions of a complex variable,

$$z' = \frac{az + b}{az + d}(ad - bc \neq 0), \qquad z' = \frac{a\bar{z} + c}{cz + d}(ad - bc \neq 0),$$

$$z' = az + b(a \neq 0), \qquad z' = a\bar{z} + b(a \neq 0),$$

embraces the isometric transformations of the first and second kind ($z' = az + b$, $z' = \overline{az} + b$, where $|a| = 1$), similarity transformations of the first and second kind ($z' = az + b$, $z' = \bar{a}\bar{z} + b, a \neq 0$) and circular transformations (the case of a linear fractional function; in particular, the inversion $z' = a/z$).

Chapter IV gives a survery of the properties of inversion of a plane and space and various applications (inversors, the geometry of Mascheroni, and the mapping of regions under inversion). In particular, detailed consideration is given to various stereographic projections of a sphere onto a plane and the construction of conformal maps of a spectrum of meridians and parallels of the sphere.

The final chapter, Chapter V, contains a list of basic definitions, formulas and the bibliography. The bibliography contains books in which the reader will find proofs of the formulas used in this text; they include textbooks on vector algebra, analytic geometry, the theory of geometric transformations, and the theory of functions of a complex variable.

The general methods for solving geometric problems described in this text are closely interrelated: it will be recalled that vector algebra is closely related with analytic geometry. The basic formulas used in Chapter III are derived on the basis of facts taken from analytic geometry; the linear fractional function of a complex variable contains within it the inversion transformation; the inversion transformation can be reliably studied by the methods of analytic geometry, and so forth.

I would like to point out that the drawing on the cover of the book (it is the same as that in Fig. 114) is a copy of a *photograph* of a model that I constructed to illustrate the stereographic projection of a sphere onto a plane under which the parallels and meridians pass into a hyperbolic pencil of circles and an associated elliptical pencil of circles. Figures 107 and 108 were done in the same manner.

During the writing of this text I received valuable advice from Professor V. A. Ilin and Corresponding Member of the USSR Academy of Sciences S. V. Yablonsky, to whom I express my deep gratitude. Very profound and valuable advice was obtained from the reviewer of the Nauka Publishing House; practically all his suggestions were incorporated in the final version of the manuscript.

It goes without saying that the general methods of solving elementary-geometry problems given in this text do not exhaust the range of such methods. For instance, mention may be made of a very powerful analytic method for applying trilinear coordinates in the plane, and tetrahedral coordinates in space (the trilinear coordinates of a point on a projective-Euclidean plane are the projective coordinates of proper points of such a plane, provided that all four fundamental points of the projective system of coordinates are also proper points; the same goes for space as well). The limited scope of this book did not allow for the inclusion of that method. And there are of course other general methods, which, unfortunately, have not been discussed in our textbooks or teaching literature (for example, synthetic methods of solving problems with the use of isometric, similarity, affine and projective transformations). However, I am sure that this situation will be remedied in time.

P. S. Modenov

The present edition was prepared after the author died. The material of the book has been re-examined and brought into accord with generally accepted terminology and notation. A small number of inaccuracies in the Russian edition have been corrected and the bibliography has been expanded.

Chapter I

VECTOR ALGEBRA

Sec. 1. Vectors in the plane (solved problems)

Problem 1. Given the angles A, B, C of $\triangle ABC$. Find $\angle\varphi = \angle BAM$, where M is the midpoint of BC.

Solution.

$$\overrightarrow{AM}\uparrow\uparrow(\overrightarrow{AB}+\overrightarrow{AC})$$

and so

$$\cos\varphi = \frac{\overrightarrow{AB}(\overrightarrow{AB}+\overrightarrow{AC})}{|\overrightarrow{AB}|\,|\overrightarrow{AB}+\overrightarrow{AC}|} = \frac{AB^2+\overrightarrow{AB}\cdot\overrightarrow{AC}}{c\sqrt{c^2+b^2+2bc\cos A}}$$

$$= \frac{c+b\cos A}{\sqrt{b^2+c^2+2bc\cos A}},$$

and since $b:c = \sin B : \sin C$, it follows that

$$\cos\varphi = \frac{\sin C + \sin B\sin A}{\sqrt{\sin^2 B + \sin^2 C + 2\sin B\sin C\cos A}}.$$

Problem 2. Given the angles A, B, C of $\triangle ABC$.
Let M be the midpoint of segment AB, and let D be the foot of the bisector of $\angle C$. Find the ratio (CDM): (ABC) and also $\varphi = \angle DCM$.

Solution.

$$\overrightarrow{CD} = \frac{a\mathbf{b}+b\mathbf{a}}{a+b},\quad \overrightarrow{CM} = \frac{\mathbf{a}+\mathbf{b}}{2},$$

where $\mathbf{a} = \overrightarrow{CB}$, $\mathbf{b} = \overrightarrow{CA}$. Consequently

$$(CDM) = \frac{1}{2}(\overrightarrow{CD},\ \overrightarrow{CM}) = \frac{(a\mathbf{b}+b\mathbf{a},\ \mathbf{a}+\mathbf{b})}{4(a+b)}$$

$$= \frac{(b-a)(\mathbf{a},\ \mathbf{b})}{4(a+b)} = \frac{(a-b)(ABC)}{2(a+b)}$$

whence

$$\frac{(CDN)}{(ABC)} = \frac{a-b}{2(a+b)},$$

which can also be written as

$$\frac{(CDM)}{(ABC)} = \frac{\sin A - \sin B}{2(\sin A + \sin B)}.$$

Furthermore, since $\overrightarrow{CD} \uparrow\uparrow (a\mathbf{b} + b\mathbf{a})$, $\overrightarrow{CM} \uparrow\uparrow (\mathbf{a} + \mathbf{b})$, it follows that

$$\cos\varphi = \frac{(a\mathbf{b} + b\mathbf{a})(\mathbf{a} + \mathbf{b})}{|a\mathbf{b} + b\mathbf{a}|\,|\mathbf{a} + \mathbf{b}|}$$

$$= \frac{ab^2 + ba^2 + ab(a + b)\cos C}{\sqrt{2a^2b^2 + 2a^2b^2\cos C}\,\sqrt{a^2 + b^2 + 2ab\cos C}}$$

$$= \frac{(a + b)\cos(C/2)}{\sqrt{a^2 + b^2 + 2ab\cos C}} = \frac{(\sin A + \sin B)\cos(C/2)}{\sqrt{\sin^2 A + \sin^2 B + 2\sin A\sin B\cos C}},$$

$$\sin\varphi = \frac{(a\mathbf{b} + b\mathbf{a},\ \mathbf{a} + \mathbf{b})}{|a\mathbf{b} + b\mathbf{a}|\,|\mathbf{a} + \mathbf{b}|} = \frac{(b - a)(\mathbf{a}, \mathbf{b})}{ab\sqrt{2(1 + \cos C)}\,\sqrt{a^2 + b^2 + 2ab\cos C}}$$

$$= \frac{(\sin B - \sin A)\sin(C/2)}{\sqrt{\sin^2 A + \sin^2 B + 2\sin A\sin B\cos C}}.$$

Problem 3. Given the interior angles A, B, C of $\triangle ABC$; M is the midpoint of segment BC, N is the foot of the altitude dropped from point C to side AB, and O is the point of intersection of the straight lines AM and CN. Find $\cos\varphi$, where $\varphi = \angle AOC$.

Solution. Orientate the plane with the base $\mathbf{a}$, $\mathbf{b}$, where $\mathbf{a} = \overrightarrow{CB}$, $\mathbf{b} = \overrightarrow{CA}$. Then $\overrightarrow{CN} \uparrow\uparrow [\mathbf{a} - \mathbf{b}]$. Indeed, the vector $[\mathbf{a} - \mathbf{b}] = [\overrightarrow{BA}]$ is perpendicular to the straight line AB and forms acute angles with the vectors $\mathbf{a}$ and $\mathbf{b}$ since

$$[\mathbf{a} - \mathbf{b}]\,\mathbf{a} = -[\mathbf{b}]\,\mathbf{a} = (\mathbf{a}, \mathbf{b}) > 0,$$

$$[\mathbf{a} - \mathbf{b}]\,\mathbf{b} = [\mathbf{a}]\,\mathbf{b} = (\mathbf{a}, \mathbf{b}) > 0.$$

Furthermore, $\overrightarrow{AM} \uparrow\uparrow \frac{\mathbf{a}}{2} - \mathbf{b} \uparrow\uparrow \mathbf{a} - 2\mathbf{b}$.

The desired $\angle\varphi$ is the angle between the vectors $\overrightarrow{AM}$ and $\overrightarrow{CN}$; consequently,

$$\cos\varphi = \frac{[\mathbf{a} - \mathbf{b}](\mathbf{a} - 2\mathbf{b})}{|[\mathbf{a} - \mathbf{b}]|\,|\mathbf{a} - 2\mathbf{b}|} = \frac{-2[\mathbf{a}]\mathbf{b} - [\mathbf{b}]\mathbf{a}}{|\mathbf{a} - \mathbf{b}|\,|\mathbf{a} - 2\mathbf{b}|} =$$

$$= \frac{-2(\mathbf{a}, \mathbf{b}) + (\mathbf{a}, \mathbf{b})}{|\mathbf{a} - \mathbf{b}|\,|\mathbf{a} - 2\mathbf{b}|} = -\frac{(\mathbf{a}, \mathbf{b})}{|\mathbf{a} - \mathbf{b}|\,|\mathbf{a} - 2\mathbf{b}|}$$

$$= -\frac{ab\sin C}{\sqrt{a^2+b^2-2ab\cos C}\,\sqrt{a^2+4b^2-4ab\cos C}}$$

$$= -\frac{\sin A\sin B\sin C}{\sqrt{\sin^2 A+\sin^2 B-2\sin A\sin B\cos C}\sqrt{\sin^2 A+4\sin^2 B-4\sin A\sin B\cos C}}.$$

Problem 4. Given vectors $\mathbf{a} = \overrightarrow{CB}$, $\mathbf{b} = \overrightarrow{CA}$. Find the vector $\mathbf{x} = \overrightarrow{CO}$, where O is the centre of a circle circumscribed about $\triangle ABC$.

Solution. From the relations

$$\mathbf{x}^2 = (\mathbf{a}-\mathbf{x})^2 = (\mathbf{b}-\mathbf{x})^2$$

we find

$$\mathbf{xa} = a^2/2, \quad \mathbf{xb} = b^2/2$$

and, hence, by the Gibbs formula

$$\mathbf{x} = \mathbf{xa}\cdot\mathbf{a}^* + \mathbf{xb}\cdot\mathbf{b}^* = \frac{a^2}{2}\mathbf{a}^* + \frac{b^2}{2}\mathbf{b}^*$$

where $\mathbf{a}^*$, $\mathbf{b}^*$ is the reciprocal (or dual) basis of the basis $\mathbf{a}$, $\mathbf{b}$:

$$\mathbf{a}^* = \frac{[\mathbf{b}]}{(\mathbf{b}, \mathbf{a})}, \quad \mathbf{b}^* = \frac{[\mathbf{a}]}{(\mathbf{a}, \mathbf{b})}.$$

Thus,

$$\mathbf{x} = \frac{b^2[\mathbf{a}] - a^2[\mathbf{b}]}{(\mathbf{a}, \mathbf{b})}.$$

Remark. If we take $\mathbf{x}$ in the form $\mathbf{x} = \lambda\mathbf{a} + \mu\mathbf{b}$, then from the relations

$$\mathbf{xa} = a^2/2, \quad \mathbf{xb} = b^2/2$$

we obtain

$$\lambda a^2 + \mu\cdot\mathbf{ab} = a^2/2,$$
$$\lambda\cdot\mathbf{ab} + \mu b^2 = b^2/2,$$

whence

$$\lambda = \frac{a^2b^2 - b^2\cdot\mathbf{ab}}{2(a^2b^2-(\mathbf{ab})^2)}, \quad \mu = \frac{a^2b^2 - a^2\cdot\mathbf{ab}}{2(a^2b^2-(\mathbf{ab})^2)}$$

and so

$$\mathbf{x} = \frac{1}{2}\,\frac{a^2b^2 - b^2\cdot\mathbf{ab}}{a^2b^2-(\mathbf{ab})^2}\,\mathbf{a} + \frac{1}{2}\,\frac{a^2b^2 - a^2\cdot\mathbf{ab}}{a^2b^2-(\mathbf{ab})^2}\,\mathbf{b}.$$

Problem 5. Given the cross products $(\mathbf{a}, \mathbf{x}) = p$, $(\mathbf{b}, \mathbf{x}) = q$ of vector $\mathbf{x}$ into the noncollinear vectors $\mathbf{a}$ and $\mathbf{b}$. Express the vector $\mathbf{x}$ in terms of the vectors $\mathbf{a}$, $\mathbf{b}$ and the numbers p, q.

Solution. Let

$$\mathbf{a}^* = \frac{[\mathbf{b}]}{(\mathbf{b},\ \mathbf{a})},\quad \mathbf{b}^* = \frac{[\mathbf{a}]}{(\mathbf{a},\ \mathbf{b})}$$

be the reciprocal basis of **a**, **b**. Then

$$\mathbf{x} = (\mathbf{x}\mathbf{a}^*)\mathbf{a} + (\mathbf{x}\mathbf{b}^*)\mathbf{b} = \left(\frac{[\mathbf{b}]}{(\mathbf{b},\ \mathbf{a})},\ \mathbf{x}\right)\mathbf{a} + \left(\frac{[\mathbf{a}]}{(\mathbf{a},\ \mathbf{b})},\ \mathbf{x}\right)\mathbf{b}$$

$$= \frac{(\mathbf{b},\ \mathbf{x})}{(\mathbf{b},\ \mathbf{a})}\mathbf{a} + \frac{(\mathbf{a},\ \mathbf{x})}{(\mathbf{a},\ \mathbf{b})}\mathbf{b} = \frac{q}{(\mathbf{b},\ \mathbf{a})}\mathbf{a} + \frac{p}{(\mathbf{a},\ \mathbf{b})}\mathbf{b} = \frac{p\mathbf{b} - q\mathbf{a}}{(\mathbf{a},\ \mathbf{b})}.$$

Problem 6. Two forces $\mathbf{F}_1 = \{2, 3\}$ and $\mathbf{F}_2 = \{4, 1\}$ are specified relative to a general Cartesian system of coordinates. Their points of application are, respectively, $A = (1,1)$ and $B = (2, 4)$. Find the coordinates of the resultant and the equation of the straight line l containing it.

Solution. The coordinates of the resultant **F** are 6 and 4. Now let $M(x, y)$ be an arbitrary point of l. Then the moment of the resultant about point M is equal to zero. This moment is equal to the sum of the moments $(\overrightarrow{MA}, \mathbf{F}_1)$ and $(\overrightarrow{MB}, \mathbf{F}_2)$ of component forces (the cross product of vectors is distributive).

Since $\overrightarrow{MA} = \{1 - x,\ 1 - y\}$, $\overrightarrow{MB} = \{2 - x,\ 4 - y\}$, it follows that

$$(\overrightarrow{MA}, \mathbf{F}_1) = \sqrt{g}\begin{vmatrix} 1 - x & 2 \\ 1 - y & 3 \end{vmatrix},\quad (\overrightarrow{MB}, \mathbf{F}_2) = \sqrt{g}\begin{vmatrix} 2 - x & 4 \\ 4 - y & 1 \end{vmatrix}$$

and, hence, the equation of the straight line l is

$$\begin{vmatrix} 1 - x & 2 \\ 1 - y & 3 \end{vmatrix} + \begin{vmatrix} 2 - x & 4 \\ 4 - y & 1 \end{vmatrix} = 0$$

or

$$4x - 6y + 13 = 0.$$

Sec. 2. Vectors in space (solved problems)

Problem 1. The plane angles of a trihedral angle $OABC$ are $a = \angle BOC$, $b = \angle COA$, $c = \angle AOB$. The interior dihedral angles of the given trihedral angle are:

$$A = B(OA)C,\quad B = C(OB)A,\quad C = A(OC)B^*$$

A trihedral angle $OA^*B^*C^*$ that is the *reciprocal* of the trihedral angle $OABC$ is a trihedral angle constructed in the following manner: ray OA^*

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* The symbol $B(OA)C$ is used to denote a dihedral angle with edge OA, in the half-planes of which are points B and C.

is perpendicular to the rays OB and OC and forms an acute angle with ray OA. The rays OB^* and OC^* are constructed in similar fashion.

Let a^*, b^*, c^* be the plane angles of the trihedral angle $OA^*B^*C^*$ and let A^*, B^*, C^*, be its interior dihedral angles.

1°. Knowing, a, b, c, find $\cos A$, $\cos B$, $\cos C$.

2°. Prove that $a^* = \pi - A$, $b^* = \pi - B$, $c^* = \pi - C$.

3°. Prove that $A^* = \pi - a$, $B^* = \pi - b$, $C^* = \pi - c$.

4°. Knowing A, B, C, find $\cos a$, $\cos b$, $\cos c$.

5°. Prove that

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} = \frac{\Delta}{\sin a \sin b \sin c}$$

where

$$\Delta = \left(\begin{vmatrix} 1 & \cos b & \cos c \\ \cos b & 1 & \cos a \\ \cos c & \cos a & 1 \end{vmatrix} \right)^{1/2}$$
$$= \sqrt{1 + 2\cos a \cos b \cos c - \cos^2 a - \cos^2 b - \cos^2 c}$$

(this relation is called the *theorem of sines for a trihedral angle* $OABC$)[1),]

6°. Prove that the sine theorem for the trihedral angle $OABC$ (see item 5°) may be written in the form

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} = \frac{\Delta^*}{\Delta}$$

where

$$\Delta^* = \sqrt{1 + 2\cos a^* \cos b^* \cos c^* - \cos^2 a^* - \cos^2 b^* - \cos^2 c^*}$$
$$= \sqrt{1 - 2\cos A \cos B \cos C - \cos^2 A - \cos^2 B - \cos^2 C}.$$

Solution. 1°. Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be the direction vectors of the rays $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ ($\mathbf{e}_1 \uparrow\uparrow \overrightarrow{OA}$, $\mathbf{e}_2 \uparrow\uparrow \overrightarrow{OB}$, $\mathbf{e}_3 \uparrow\uparrow \overrightarrow{OC}$). Then the vectors $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$ of the reciprocal basis of the basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are the direction vectors of the rays $\overrightarrow{OA^*}$, $\overrightarrow{OB^*}$, $\overrightarrow{OC^*}$. We assume the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ to be unit vectors and lay them off from point O; then their endpoints E_1, E_2, E_3 will lie on the ray $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ respectively. Through O draw a plane π perpendicular to the ray $\overrightarrow{OC}$. Let E_1^0 and E_2^0 be orthogonal projections of the points E_1 and E_2 on the plane π. Then $C = A(OC)B = \angle E_1^0 O E_2^0$. Consider the vectors

$$\mathbf{e}_1^0 = \overrightarrow{OE_1^0}, \quad \mathbf{e}_2^0 = \overrightarrow{OE_2^0}.$$

---

1) Or for a *spherical triangle* cut out of a sphere with centre O by a trihedral angle.

We have

$$\mathbf{e}_1^0 = \mathbf{e}_1 + \lambda \mathbf{e}_3, \quad \mathbf{e}_2^0 = \mathbf{e}_2 + \mu \mathbf{e}_3.$$

Forming the scalar product of both sides of each of these relations by the vector $\mathbf{e}_3$, we obtain

$$0 = \cos b + \lambda, \quad 0 = \cos a + \mu,$$

so that

$$\mathbf{e}_1^0 = \mathbf{e}_1 - \mathbf{e}_3 \cos b, \quad \mathbf{e}_2^0 = \mathbf{e}_2 - \mathbf{e}_3 \cos a$$

and consequently

$$\cos C = \frac{\mathbf{e}_1^0 \cdot \mathbf{e}_2^0}{|\mathbf{e}_1^0|\,|\mathbf{e}_2^0|} = \frac{(\mathbf{e}_1 - \mathbf{e}_3 \cos b)(\mathbf{e}_2 - \mathbf{e}_3 \cos a)}{\sqrt{(\mathbf{e}_1 - \mathbf{e}_3 \cos b)^2}\,\sqrt{(\mathbf{e}_2 - \mathbf{e}_3 \cos a)^2}}$$

$$= \frac{\cos c - \cos b \cos a - \cos a \cos b + \cos b \cos a}{\sqrt{1 - 2\cos^2 b + \cos^2 b}\,\sqrt{1 - 2\cos^2 a + \cos^2 a}}$$

$$= \frac{\cos c - \cos a \cos b}{\sin a \sin b}.$$

In similar fashion we calculate $\cos A$ and $\cos B$. Thus,

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c},$$

$$\cos B = \frac{\cos b - \cos c \cos a}{\sin c \sin a}$$

$$\cos C = \frac{\cos c - \cos a \cos b}{\sin a \sin b}.$$

2°. The formulas obtained in item 1° can be rewritten thus:

$$\cos A = \frac{[\mathbf{e}_1\mathbf{e}_2] \cdot [\mathbf{e}_1\mathbf{e}_3]}{|[\mathbf{e}_1\mathbf{e}_2]|\,|[\mathbf{e}_1\mathbf{e}_3]|},$$

$$\cos B = \frac{[\mathbf{e}_2\mathbf{e}_1] \cdot [\mathbf{e}_2\mathbf{e}_3]}{|[\mathbf{e}_2\mathbf{e}_1]|\,|[\mathbf{e}_2\mathbf{e}_3]|},$$

$$\cos C = \frac{[\mathbf{e}_3\mathbf{e}_1] \cdot [\mathbf{e}_3\mathbf{e}_2]}{|[\mathbf{e}_3\mathbf{e}_1]|\,|[\mathbf{e}_3\mathbf{e}_2]|}.$$

Note that in this notation the vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ need not necessarily be regarded as unit vectors because when $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are replaced respectively by $\lambda \mathbf{e}_1, \mu \mathbf{e}_2, \nu \mathbf{e}_3$, where $\lambda > 0$, $\mu > 0$, $\nu > 0$, the right-hand members of these relations remain unchanged. Thus, $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ may be regarded as any direction vectors of the rays $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$.