Lecture Notes in Mathematics

Catherine Donati-Martin
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Alain Rouault
(Eds.)

Séminaire de Probabilités XLIII

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Preface

The series of advanced courses, initiated in Séminaire de Probabilités XXXIII, continues with a course of Jean Picard on the representation formulae for the fractional Brownian motion. The rest of the volume covers a wide range of themes, such as stochastic calculus and stochastic differential equations, stochastic differential geometry, filtrations, analysis on Wiener space, random matrices and free probability, mathematical finance. Some of the contributions were presented at the Journées de Probabilités held in Poitiers in June 2009.

The Séminaire has now a new web site at the URL

`http://portail.mathdoc.fr/SemProba/`

This web site is hosted by the Cellule Math Doc funded both by the CNRS and the Université Joseph Fourier in Grenoble, France. We thank the team of the Institut de Recherche Mathématiques Avancées (IRMA) in Strasbourg for the maintenance of the former web site.

With the new web site also comes a new multicriteria research tool which improves the previous one. This tool has been developped by the Cellule MathDoc (Laurent Guillopé, Elizabeth Cherhal and Claude Goutorbe). The enormous work of indexing and commenting was started by Paul-André Meyer in 1995 with the help of other editors, with an important contribution from Michel Émery (who performed the supervision of all the work) and Marc Yor. The database covers now the contents of volumes I to XL. We expect to complete the work soon in order to provide some easy way to exploit fully the content of the Séminaire.

We remind you that the Cellule Math Doc also hosts digitized articles of many scientific journals within the NUMDAM project. All the articles of the Seminaire from Volume I in 1967 to Volume XXXVI in 2002 are freely accessible from this web site

`http://www.numdam.org/numdam-bin/feuilleter?j=SPS`

Finally, the Rédaction of the Séminaire is modified: Christophe Stricker and Michel Émery retired from our team after Séminaire XLII was completed. Both contributed early and continuously as authors and accepted to invest energy and time as Rédacteurs. Michel Émery was a member of the board since volume XXIX.

During all these years, the Séminaire benefited from his demanding quality requirements, be it on mathematics and on style. His meticulous reading of articles was sometimes supplemented by a rewriting suggesting notably elegant phrases instead of basic English.

While preparing this volume, we heard the sad news that Lester Dubins, professor emeritus at Berkeley University, passed away. From the early days, several talentuous mathematicians from various countries have contributed to the Séminaire and Dubins was one of them.

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Contents

Specialized Course

Representation Formulae for the Fractional Brownian Motion

Jean Picard

Abstract We discuss the relationships between some classical representations of the fractional Brownian motion, as a stochastic integral with respect to a standard Brownian motion, or as a series of functions with independent Gaussian coefficients. The basic notions of fractional calculus which are needed for the study are introduced. As an application, we also prove some properties of the Cameron–Martin space of the fractional Brownian motion, and compare its law with the law of some of its variants. Several of the results which are given here are not new; our aim is to provide a unified treatment of some previous literature, and to give alternative proofs and additional results; we also try to be as self-contained as possible.

Keywords Fractional Brownian motion · Cameron-Martin space · Laws of Gaussian processes

1 Introduction

Consider a fractional Brownian motion $(B_t^H; t \in \mathbb{R})$ with Hurst parameter $0 < H < 1$. These processes appeared in 1940 in [24], and they generalise the case $H = 1/2$ which is the standard Brownian motion. A huge literature has been devoted to them since the late 1960s. They are often used to model systems involving Gaussian noise, but which are not correctly explained with a standard Brownian motion. Our aim here is to give a few basic results about them, and in particular to explain how all of them can be deduced from a standard Brownian motion.

The process B^H is a centred Gaussian process which has stationary increments and is H-self-similar; these two conditions can be written as

$$B_{t+t_0}^H - B_{t_0}^H \simeq B_t^H, \qquad B_{\lambda t}^H \simeq \lambda^H B_t^H \tag{1}$$

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for $t_0 \in \mathbb{R}$ and $\lambda > 0$, where the notation $Z_t^1 \simeq Z_t^2$ means that the two processes have the same finite dimensional distributions. We can deduce from (1) that B_{-t}^H and B_t^H have the same variance, that this variance is proportional to $|t|^{2H}$, and that the covariance kernel of B^H must be of the form

$$\begin{aligned} C(s,t) = \mathbb{E}\big[B_s^H B_t^H\big] &= \frac{1}{2}\mathbb{E}\Big[(B_s^H)^2 + (B_t^H)^2 - (B_t^H - B_s^H)^2\Big] \\ &= \frac{1}{2}\mathbb{E}\Big[(B_s^H)^2 + (B_t^H)^2 - (B_{t-s}^H)^2\Big] \\ &= \frac{\rho}{2}\Big(|s|^{2H} + |t|^{2H} - |t-s|^{2H}\Big) \end{aligned} \tag{2}$$

for a positive parameter $\rho = \mathbb{E}[(B_1^H)^2]$ (we always assume that $\rho \neq 0$). The process B^H has a continuous modification (we always choose this modification), and its law is characterised by the two parameters ρ and H; however, the important parameter is H, and ρ is easily modified by multiplying B^H by a constant. In this article, it will be convenient to suppose $\rho = \rho(H)$ given in (51); this choice corresponds to the representation of B^H given in (6). We also consider the restriction of B^H to intervals of $\mathbb{R}$ such as $\mathbb{R}_+$, $\mathbb{R}_-$ or $[0,1]$.

Notice that the fractional Brownian motion also exists for $H = 1$ and satisfies $B_t^1 = t\, B_1^1$; this is however a very particular process which is excluded from our study (with our choice of $\rho(H)$ we have $\rho(1) = \infty$).

The standard Brownian motion $W_t = B_t^{1/2}$ is the process corresponding to $H = 1/2$ and $\rho = \rho(1/2) = 1$. It is often useful to represent B^H for $0 < H < 1$ as a linear functional of W; this means that one looks for a kernel $K^H(t,s)$ such that the Wiener-Itô integral

$$B_t^H = \int K^H(t,s)dW_s \tag{3}$$

is a H-fractional Brownian motion. More generally, considering the family $(B^H;\ 0 < H < 1)$ defined by (3), we would like to find $K^{J,H}$ so that

$$B_t^H = \int K^{J,H}(t,s)dB_s^J. \tag{4}$$

In this case however, we have to give a sense to the integral; the process B^J is a Gaussian process but is not a semimartingale for $J \neq 1/2$, so we cannot consider Itô integration. In order to solve this issue, we approximate B^J with smooth functions for which the Lebesgue–Stieltjes integral can be defined, and then verify that we can pass to the limit in an adequate functional space in which B^J lives almost surely. Alternatively, it is also possible to use integration by parts.

The case where $K^{J,H}$ is a Volterra kernel ($K^{J,H}(t,s) = 0$ if $s > t$) is of particular interest; in this case, the completed filtrations of B^H and of the increments of B^J satisfy $\mathcal{F}_t(B^H) \subset \mathcal{F}_t(dB^J)$, with the notation

$$\mathcal{F}_t(X) = \sigma(X_s; s \le t), \quad \mathcal{F}_t(dX) = \sigma(X_s - X_u; u \le s \le t). \tag{5}$$

Notice that when the time interval is $\mathbb{R}_+$, then $\mathcal{F}_t(dB^J) = \mathcal{F}_t(B^J)$ (because $B_0^J = 0$), but this is false for $t < 0$ when the time interval is $\mathbb{R}$ or $\mathbb{R}_-$. When $\mathcal{F}_t(B^H) = \mathcal{F}_t(B^J)$, we say that the representation (4) is canonical; actually, we extend here a terminology, introduced by [25] (see [16]), which classically describes representations with respect to processes with independent increments (so here the representation (3)); such a canonical representation is in some sense unique.

Another purpose of this article is to compare B^H with two other families of processes with similar properties and which are easier to handle in some situations:

- The so-called Riemann–Liouville processes on $\mathbb{R}_+$ (they are also sometimes called type II fractional Brownian motions, see [27]), are deduced from the standard Brownian motion by applying Riemann–Liouville fractional operators, whereas, as we shall recall it, the genuine fractional Brownian motion requires a weighted fractional operator.
- We shall also consider here some processes defined by means of a Fourier-Wiener series on a finite time interval; they are easy to handle in Fourier analysis, whereas the Fourier coefficients of the genuine fractional Brownian motion do not satisfy good independence properties.

We shall prove that the Cameron–Martin spaces of all these processes are equivalent, and we shall compare their laws; more precisely, it is known from [10, 15, 16] that two Gaussian measures are either equivalent, or mutually singular, and we shall decide between these two possibilities.

Let us now describe the contents of this article. Notations and definitions which are used throughout the article are given in Sect. 2; we also give in this section a short review of fractional calculus, in particular Riemann–Liouville operators and some of their modifications which are important for our study; we introduce some functional spaces of Hölder continuous functions; much more results can be found in [35]. In Sect. 3, we give some results concerning the time inversion ($t \mapsto 1/t$) of Gaussian self-similar processes.

We enter the main topic in Sect. 4. Our first aim is to explore the relationship between two classical representations of B^H with respect to W, namely the representation of [26],

$$B_t^H = \frac{1}{\Gamma(H+1/2)} \int_{\mathbb{R}} \left((t-s)_+^{H-1/2} - (-s)_+^{H-1/2} \right) dW_s \tag{6}$$

on $\mathbb{R}$ (with the notation $u_+^\lambda = u^\lambda 1_{\{u>0\}}$), and the canonical representation on $\mathbb{R}_+$ obtained in [29, 30], see also [8, 32] (this is a representation of type (3) for a Volterra kernel K^H, and such that W and B^H generate the same filtration). Let us explain the idea by means of which this relationship can be obtained; in the canonical representation on $\mathbb{R}_+$, we want B_t^H to depend on past values W_s, $s \leq t$, or equivalently, we want the infinitesimal increment dB_t^H to depend on past increments dW_s, $s \leq t$. In (6), values of B_t^H for $t \geq 0$ involve values of W_s for all $-\infty \leq s \leq t$, so this is not convenient for a study on $\mathbb{R}_+$. However, we can reverse the time ($t \mapsto -t$) and use the backward representation

$$B_t^H = \frac{1}{\Gamma(H+1/2)} \int_0^{+\infty} \left(s^{H-1/2} - (s-t)_+^{H-1/2} \right) dW_s$$

on $\mathbb{R}_+$. Now the value of B_t^H involves the whole path of W on $\mathbb{R}_+$, but we can notice that the infinitesimal increment dB_t^H only involves future increments dW_s, $s \geq t$. Thus $dB^H(1/t)$ depends on past increments $dW(1/s)$, $s \leq t$. We can then conclude by applying the invariance of fractional Brownian motions by time inversion which has been proved in Sect. 3. This argument is justified in [29] by using the generalised processes dB_t^H/dt, but we shall avoid the explicit use of these processes here. This technique can be used to work out a general relationship of type (4) between B^H and B^J for any $0 < J, H < 1$, see Theorem 11 (such a relation was obtained by [20]).

Application of time inversion techniques also enables us to deduce in Theorem 13 a canonical representation on $\mathbb{R}_-$, and to obtain in Theorem 16 some non canonical representations of B^H with respect to itself, extending the classical case $H = 1/2$; these representations are also considered by [21].

Representations of type (3) or (4) can be applied to descriptions of the Cameron–Martin spaces $\mathcal{H}_H$ of the fractional Brownian motions B^H; these spaces are Hilbert spaces which characterise the laws of centred Gaussian processes (see Appendix C). The space $\mathcal{H}_{1/2}$ is the classical space of absolutely continuous functions h such that $h(0) = 0$ and the derivative $D^1 h$ is square integrable, and (3) implies that $\mathcal{H}_H$ is the space of functions of the form

$$t \mapsto \frac{1}{\Gamma(H+1/2)} \int_{\mathbb{R}} \left((t-s)_+^{H-1/2} - (-s)_+^{H-1/2} \right) f(s)ds$$

for square integrable functions f.

Sections 5 and 6 are devoted to the comparison of B^H with two processes. One of them is self-similar but has only asymptotically stationary increments in large time, and the other one has stationary increments, but is only asymptotically self-similar in small time.

In Sect. 5, we consider on $\mathbb{R}_+$ the so-called Riemann–Liouville process defined for $H > 0$ by

$$X_t^H = \frac{1}{\Gamma(H+1/2)} \int_0^t (t-s)^{H-1/2} dW_s.$$

This process is H-self-similar but does not have stationary increments; contrary to B^H, the parameter H can be larger than 1. The Cameron–Martin space $\mathcal{H}'_H$ of X^H is the space of functions

$$t \mapsto \frac{1}{\Gamma(H+1/2)} \int_0^t (t-s)^{H-1/2} f(s)ds$$

for square integrable functions f. We explain in Theorem 19 a result of [35], see [8], stating that $\mathcal{H}_H$ and $\mathcal{H}'_H$ are equivalent for $0 < H < 1$ (they are the same set with equivalent norms). We also compare the paths of B^H and X^H, and in particular