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Takuro Mochizuki

Donaldson Type Invariants for Algebraic Surfaces

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Transition of Moduli Stacks

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To my friends Yoshinobu Akahori
and Shu Kawaguchi for memory
of our informal student seminar

Preface

In this monograph, we define and investigate an algebro-geometric analogue of Donaldson invariants by using moduli spaces of semistable sheaves with arbitrary ranks on a polarized projective surface. We may expect the existence of interesting "universal relations among invariants", which would be a natural generalization of the "wall-crossing formula" and the "Witten conjecture" for classical Donaldson invariants. Our goal is to obtain a weaker version of such relations, in other brief words, to describe a relation as the sum of integrals over the products of moduli spaces of objects with lower ranks. Fortunately, according to a recent excellent work of L. Göttsche, H. Nakajima and K. Yoshioka, [53], a wall-crossing formula for Donaldson invariants of projective surfaces can be deduced from such a weaker result in the rank two case.

We hope that our work in this monograph would, at least tentatively, provides a part of foundation for the further study on such universal relations.

In the rest of this preface, we would like to explain our motivation and some of important ingredients of this study. See *Introduction* for our actual problems and results.

Donaldson Invariants

Let us briefly recall Donaldson invariants. We refer to [22] for more details and precise. We also refer to [37], [39], [51] and [53]. Let X be a compact simply connected oriented real 4-dimensional C^∞-manifold with a Riemannian metric g. Let P be a principal $SO(3)$-bundle on X. A connection ∇ of P is called an anti-self-dual (ASD) connection if the curvature $F(\nabla)$ satisfies $*_g F(\nabla) = -F(\nabla)$, where $*_g$ denotes the Hodge star operator associated to g. For simplicity, let us restrict ourselves to the case that P comes from a principal $SU(2)$-bundle. Note that P is determined by its second Chern class $c_2 := c_2(P)$. Let $b^+(X)$ be the number of positive eigenvalues of the intersection form on $H^2(X)$. Let $M(c_2, g)$ denote the moduli space of ASD connections of P. If g is sufficiently general, it is known that $M(c_2, g)$ is naturally a C^∞-manifold with $\dim_{\mathbb{R}} M(c_2, g) = 8c_2 - 3\big(1 + b^+(X)\big)$.

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Let $d := \dim_{\mathbb{R}} M(c_2, g)/2$ and write $d = l + 2m$. We consider integrals

$$\Phi_d^{X,g}(\alpha_1 \cdot \cdots \cdot \alpha_l \cdot p^m) := \int_{M(c_2,g)} \mu(\alpha_1) \cup \cdots \cup \mu(\alpha_l) \cdot \mu(P)^m \tag{0.1}$$

for $\alpha_i \in H_2(X, \mathbb{Q})$ $(i = 1, \ldots, l)$ and $p \in H_0(X, \mathbb{Q})$. The map $\mu : H_*(X, \mathbb{Q}) \longrightarrow H^*\big(M(c_2, g), \mathbb{Q}\big)$ is formally given by $\mu(\alpha) = c_2(\mathcal{P})/\alpha$, where $\mathcal{P}$ is the universal principal bundle on $X \times M(c_2, g)$. Although $M(c_2, g)$ is not compact in general, $\mu(\alpha_i)$ are naturally extended to the cohomology classes on the Uhlenbeck compactification $\overline{M}(c_2, g)$.

Moreover, let $\mathfrak{A}_*(X)$ be the symmetric algebra generated by $H_2(X) \oplus H_0(X)$, which is graded by giving degree $2 - i/2$ to elements of $H_i(X)$. Then, $\Phi_d^{X,g}$ gives a linear map $\mathfrak{A}_d(X) \longrightarrow \mathbb{Q}$. Thus, we obtain a map

$$\Phi^{X,g} := \sum_{d \geq 0} \Phi_d^{X,g} : \mathfrak{A}_*(X) \longrightarrow \mathbb{Q}. \tag{0.2}$$

It is called a Donaldson invariant of X. If $b^+(X) > 1$, (0.1) are shown to be independent of the choice of general g.

They were successfully applied in the study of low-dimensional differential topology, until the appearance of Seiberg-Witten invariants which are defined more easily and believed to contain equivalent information in most cases. Nowadays, the attention to Donaldson invariants has been rather limited. So the author should explain why he would like to study a generalization of their algebro-geometric analogue in this comparatively long monograph. Although they might be less interesting in terms of topological application, there would exist attractive problems of "universal relations among invariants" which are natural generalizations of the "wall-crossing formula" ("Kotschick-Morgan conjecture") and the "Witten conjecture".

We remark that P. Kronheimer [76] studied such a generalization of Donaldson invariants for real 4-dimensional manifolds by using the moduli of higher rank objects from a viewpoint of differential geometry. It was also investigated in mathematical physics. (See [84], for example.)

Kotschick-Morgan Conjecture and Witten Conjecture

Let us recall the conjectures for an explanation of the motivation of our study, although they are beyond the scope of this monograph. We should remark that they have been studied intensively from mathematical viewpoints, by V. Y. Pidstrigach–A. N. Tyurin, B. Chen, and in particular P. Feehan–T. Leness.

In the case $b^+ = 1$, the integrals (0.1) depend on the choice of the metric g. Let us recall the result of D. Kotschick–J. Morgan in [74], following [51]. Let $H^2(X, \mathbb{R})^+$ denote the set of $\alpha \in H^2(X, \mathbb{R})$ such that $\alpha^2 > 0$. Then, $H^2(X, \mathbb{R})^+/\mathbb{R}^+$ has two connected components. Let us fix one component Ω^+. The period point $\omega(g)$ is the point in Ω^+ defined by the closed two form which is harmonic with respect to g and satisfies $*_g\omega(g) = \omega(g)$. An element $\xi \in H^2(X, \mathbb{Z})$ is called of type d, if

$(d+3)/4+\xi^2 \in \mathbb{Z}_{\geq 0}$. For such ξ, let $W^\xi := \{L \in \Omega^+ \mid L \cdot \xi = 0\}$ be called a wall of type d. The chambers of type d are the connected components in the complement of the walls of type d in Ω^+. Kotschick and Morgan proved that $\Phi_d^{X,g}$ depends only on the chamber in which $\omega(g)$ is contained. It is very interesting to ask what happens when $\omega(g)$ crosses a wall of type d, that is called a wall-crossing phenomenon. We obtain the linear map $\delta_{\xi,d}^X : \mathfrak{A}_d(X) \longrightarrow \mathbb{Q}$ given by $\Phi_d^{X,g_1} - \Phi_d^{X,g_2} = \delta_{\xi,d}^X$, where the chambers of $\omega(g_1)$ and $\omega(g_2)$ are divided with W^ξ. They conjectured that $\delta_{\xi,d}^X(\alpha^d)$ $(\alpha \in H_2(X))$ is the polynomial in the intersection numbers α^2 and $\alpha \cdot \xi$ whose coefficients depend only on ξ^2, d and the homotopy type of X. Very interestingly, assuming the conjecture, Göttsche showed that $\delta_{\xi,d}^X$ can be expressed by using modular forms, which is called a *wall-crossing formula*. ([50]. See also [51].)

Since the appearance of Seiberg-Witten invariants, it has been expected that Seiberg-Witten invariants and Donaldson invariants give equivalent information in most cases. Let χ and τ denote the Euler number and the signature of X, respectively. Let $m(X) = 2 + (7\chi + \tau)/4$. Let Λ denote the set of isomorphism classes of line bundles L on X such that $c_1(L)^2 = 2\chi + 3\tau$. For $L \in \Lambda$, let n_L denote the Seiberg-Witten invariant, i.e., the number of the solutions of the Seiberg-Witten equation associated to a Spin^c-structure whose determinant bundle is L. Then, Donaldson invariants and Seiberg-Witten invariants are expected to be related by the following impressive formula:

$$\Phi^X(\alpha) = 2^{m(X)} e^{\alpha\cdot\alpha/2} \sum_{L\in\Lambda} n_L \cdot e^{c_1(L)\cdot\alpha}$$

(See [130] and [21] for more precise and details.)

Both conjectures claim the existence of universal relations which are expressed in impressive ways. It might be desirable to understand them as a part of larger universal relations by extending the framework to the higher rank cases, and moreover we might dream to understand some geometry behind them.

Donaldson Invariants in Algebraic Geometry

We recall how Donaldson invariants were discussed in algebraic geometry. Let X be a smooth projective surface with an ample line bundle $\mathcal{O}_X(1)$. Let ω be a Kahler metric which represents the first Chern class of $\mathcal{O}_X(1)$.

Let E be a holomorphic vector bundle on X such that $\det(E) \simeq \mathcal{O}_X$. A hermitian metric h of E is called Hermitian-Einstein with respect to ω, if $\Lambda_\omega F(h) = 0$ is satisfied, where $F(h)$ is the curvature of the canonical unitary connection ∇_h of (E, h), and Λ_ω denotes the adjoint of the multiplication of ω. Let P be the principal $SU(2)$-bundle associated to E with a hermitian metric h. Then, the induced connection ∇_h on P is ASD, if and only if h is Hermitian-Einstein. Therefore, considering ASD-connections is equivalent to considering holomorphic vector bundles with Hermitian-Einstein metrics.

Recall a very deep correspondence between objects in differential geometry and algebraic geometry, so called the Kobayashi-Hitchin correspondence. (See [18], [19], [128].) Let E be a holomorphic vector bundle on X such that $\det(E) \simeq \mathcal{O}_X$. Then, E is called μ-stable if and only if $\deg_\omega(F) := \int_X c_1(F) \cdot \omega < 0$ for any coherent subsheaf F of E. According to the Kobayashi-Hitchin correspondence, E is μ-stable if and only if E is indecomposable and has a Hermitian-Einstein metric. Moreover, such a metric is unique up to obvious ambiguity. Thus, we obtain the bijective correspondence between μ-stable bundles and holomorphic indecomposable vector bundles with Hermitian-Einstein metrics, more strongly the homeomorphism of the moduli spaces.

Stable bundles and more general stable sheaves have been studied in algebraic geometry. A moduli space $\mathcal{M}(c_2)$ of semistable sheaves was constructed as the projective variety by D. Gieseker and M. Maruyama in the framework of geometric invariant theory, which gives a compactification of moduli of stable vector bundles, called the Gieseker-Maruyama compactification. It was known that there exists the projective morphism of the Gieseker-Maruyama compactification to the Uhlenbeck compactification. It is natural to expect that the integral (0.1) can be defined as the evaluation over $\mathcal{M}(c_2)$. It doesn't work in general, since a Kahler metric is not necessarily generic and the moduli space $\mathcal{M}(c_2)$ does not have the expected dimension. But, if c_2 is sufficiently large, it is known that $\mathcal{M}(c_2)$ has the expected dimension, and the evaluation over $\mathcal{M}(c_2)$ gives the Donaldson invariants according to J. Li and Morgan. (See [81] and [94].) By using the blow up formulas due to R. Fintushel and R. J. Stern [34], which relate the Donaldson invariants of X and the blow up $\widehat{X}$ of X in a point, the full Donaldson invariants can be defined in a purely algebro-geometric way. (See [53] for more details.) Note that we will later give a different construction of invariants in an algebro-geometric way. (They are the same, if the moduli has the expected dimension.) Each method has its own advantage.

Reduction to Sum of Integrals over the Products of Hilbert Schemes

G. Ellingsrud-L. Göttsche and R. Friedman-Z. Qin studied the wall crossing phenomena of Donaldson invariants in algebraic geometry. (See [26] and [36].) Although we omit the details here, their results briefly say that if X is a smooth projective surface, $\delta^X_{\xi,d}$ can be described as the sum of integrals of naturally induced cohomology classes over the products of Hilbert schemes of points on X, under the assumption that *the wall W^ξ is good*. Such a formula is called a *weak wall-crossing formula* in this monograph, and actually one of our main goals is to show it without any assumption on the walls by using *intrinsic smoothness* of the moduli spaces.

Hilbert Schemes of Points on a Surface

The author thinks that it is a proper goal to describe some relations among invariants as the sum of integrals over the products of Hilbert schemes, although not final.

For an explanation, we recall a brief history of the study of Hilbert schemes $X^{[n]}$ of n-points on a connected projective surface X. We refer to [49] and [98] for more details. The first important result is the irreducibility and smoothness of $X^{[n]}$. Among several important and pioneering works, one of the most impressive and famous results is a formula of Göttsche. It is irresistible to reproduce it here. Let $b_i(X)$ denote the i-th Betti numbers of X. Let $P_{X^{[n]}}(z)$ denote the Poincaré polynomial of $X^{[n]}$ in a variable z. Then, Göttsche's formula is

$$\sum_{n\geq 0}^{\infty} P_{X^{[n]}}(z)t^n = \prod_{m\geq 1} \frac{(1+z^{2m-1}t^m)^{b_1(X)}(1+z^{2m+1}t^m)^{b_3(X)}}{(1-z^{2m-2}t^m)(1-z^{2m}t^m)^{b_2(X)}(1-z^{2m+2}t^m)}.$$

Many mathematicians were attracted by the formula for deeper understanding. For example, it has been shown to hold in the level of Grothendieck group of varieties ([11], [52], [16]). In the other direction, a deep observation was given by Nakajima and I. Grojnowski ([56] and [97]), who constructed geometrically a representation of Heisenberg algebra on $\bigoplus_n H^*(X^{[n]})$.

Since then, a considerable number of studies have been conducted on Hilbert schemes of points, and hence they have been much more familiar than moduli of sheaves with higher ranks. It could be one reason why we try to express some quantity as the sum of integrals over the products of Hilbert schemes. But moreover, in principle, we may expect that some information of Hilbert schemes $X^{[n]}$ are comprehensible from that of the original surface X with universal and lucid procedures which are independent of X. The formula of Göttsche and the Nakajima-Grojnowski construction are typical. The principle was enforced by the results on cobordism classes, some integrals and others. (For example, see [27] and [80].) That is the main reason why the author believes it worthwhile to express something in terms of Hilbert schemes, like a weak wall-crossing formula in the rank two case.

Fortunately for the author, after the solution of Nekrasov's conjecture ([99], [100], [101], [102] [106]), Göttsche, Nakajima and Yoshioka successfully showed that the wall-crossing formula for Donaldson invariants of projective surface can be deduced from the weak wall-crossing formula! (See [53].) We refer to the lecture notes and the survey by Nakajima and Yoshioka for this story. ([99] and [102]).

Virtual Fundamental Classes

We should emphasize that our moderate goal, such as a weak wall-crossing formula, is already related with important, interesting and rather recent development in mathematics, so called derived algebraic geometry. In their study of the wall-crossing phenomena, Ellingsrud-Göttsche and Friedman-Qin used a sequence of flips to relate the moduli spaces corresponding to two polarizations. Their argument can work if flips are neat, which is related with the assumption that the wall is good. But, singularity may appear in general.

Singularity of moduli spaces causes difficult but attractive problems in many aspects of mathematics. Since 1990's, several mathematicians, such as P. Deligne,

V. Drinfeld and M. Kontsevich, have proposed significance of *intrinsic smoothness of moduli spaces*. (See [71].) This revolutionary idea was employed most efficiently in the Gromov-Witten theory. In constructing Gromov-Witten invariants, we need to evaluate some cohomology classes over fundamental classes of moduli stacks of stable maps. However, the moduli stacks may not be smooth and of expected dimension, and hence they do not have fundamental classes in the naive sense. As a solution of this difficulty, L. Behrend-B. Fantechi, K. Fukaya-K. Ono and J. Li-G. Tian gave constructions of *virtual fundamental classes* by using intrinsic smoothness called perfect obstruction theory or Kuranishi structure ([6], [40] and [82]). Rather recently, virtual fundamental classes have also been used in the study of Donaldson-Thomas invariants [121] for three dimensional Calabi-Yau varieties.

As for our problems, we will define our algebro-geometric analogue of Donaldson invariants by using virtual fundamental classes, which allows us to utilize intrinsic smoothness effectively in the study of transitions of moduli stacks. More specifically, we can use a technique of localization with respect to torus actions to obtain weak wall crossing formulas and similar formulas, instead of flips.

Master Space

Another important ingredient is the excellent and beautiful idea of *master space* due to M. Thaddeus [120]. Let G be a linear reductive group. Let U be a projective variety with a G-action. Let $\mathcal{L}_i$ $(i = 1, 2)$ be G-polarizations of U. Then, we have the open subset $U^{ss}(\mathcal{L}_i)$ of semistable points of U with respect to $\mathcal{L}_i$. It is interesting and significant to compare the quotient stacks $\mathcal{M}_i := U^{ss}(\mathcal{L}_i)/G$ $(i = 1, 2)$. (Since the moduli stacks of semistable sheaves have such descriptions, it is clearly related with our problem.)

For that purpose, Thaddeus introduced the idea of master space. Let us consider the G-variety $\mathrm{TH} := \mathbb{P}(\mathcal{L}_1^{-1} \oplus \mathcal{L}_2^{-1})$ on U, the projectivization of the vector bundle $\mathcal{L}_1 \oplus \mathcal{L}_2$. We have the canonical polarization $\mathcal{O}_{\mathbb{P}}(1)$ on TH. We have the canonically defined G-action on TH, and $\mathcal{O}_{\mathbb{P}}(1)$ gives the G-polarization. The set of the semistable points is denoted by TH^{ss}. Then we obtain the quotient stack $M := \mathrm{TH}^{ss}/G$. Let $\pi : \mathrm{TH}^{ss} \longrightarrow M$ denote the projection.

Let G_m denote a one dimensional torus $\operatorname{Spec} \mathbb{C}[t, t^{-1}]$. We have the G_m-action on TH given by $\rho(t) \cdot [x : y] = [t \cdot x : y]$, where $[x : y]$ denotes the homogeneous coordinate of TH along the fiber direction. Since ρ commutes with the action of G, it induces a G_m-action $\overline{\rho}$ on M.

We have the natural inclusions $\mathrm{TH}_i = \mathbb{P}(\mathcal{L}_i^{-1}) \longrightarrow \mathrm{TH}$. Because $\mathcal{O}_{\mathbb{P}}(1)_{|\,\mathrm{TH}_i} = \mathcal{L}_i$, we have $\mathrm{TH}_i^{ss} = U(\mathcal{L}_i)^{ss}$ and inclusions $\mathcal{M}_i \longrightarrow M$. Since TH_i are components of the fixed point set of the action ρ, the stacks $\mathcal{M}_i$ are the components of the fixed point set of the action $\overline{\rho}$.

The action $\overline{\rho}$ may have fixed points not contained in $\mathcal{M}_1 \cup \mathcal{M}_2$. Let $x \in \mathrm{TH}^{ss}$. In general, $\pi(x)$ is fixed by the action $\overline{\rho}$ if and only if $G_m \cdot x \subset G \cdot x$. So we may have the components of the fixed points $\pi(x)$ such that x is not fixed by ρ. Such a component is called exceptional. In a sense, the information on the difference of

the quotient stacks $\mathcal{M}_i$ $(i = 1, 2)$ is contained in the exceptional fixed point sets. As already mentioned, we will extract this information by using a G_m-localization technique in our situation.

We have one more technical remark. In the study of the transition of moduli stacks of objects with higher ranks, one of the main difficulties for the author is the appearance of objects which are decomposable into more than three components. In our argument, it causes that the master space is not Deligne-Mumford. To avoid such difficulty, we introduce enhanced master spaces, which requires a detailed and interesting investigation in controlling fixed point sets.

Further Study

This monograph is tentative and experimental, partially because of rapid and intensive development of the theory of stacks. For example, J. Lurie, B. Toën and their collaborators are rewriting the foundation of algebraic geometry. (See [124] for overview.) Their theory, so called *derived algebraic geometry* or *homotopical algebraic geometry*, seems to provide us with a powerful tool in constructing perfect obstruction theories, and hence virtual fundamental classes. But, unfortunately, it is beyond the author's ability to use their theory at this moment. He believes that it should be one of the themes in the study of stacks to make it easy to deal with the objects and the formalisms in our work. He hopes to study these problems from the viewpoint of derived algebraic geometry once more in future. (However, he also believes that we will obtain the same invariants even if we adopt another method in constructing perfect obstruction theory.)

One of the important results missing in our theory is the blow-up formula, i.e., a comparison of invariants for X and a blow up of X. The author originally intended to develop a theory "without blow up". But it seems to contain interesting problems for such a comparison. For example, see the recent attractive work of Nakajima and Yoshioka, [103] and [104].

From a perfect obstruction theories, we obtain not only virtual fundamental classes but also *virtual structure sheaves*, which seem very attractive subjects. For example, it would be very interesting to find a formula to express the Euler numbers of the tensor products of line bundles and the virtual structure sheaves. (See [28], [54] and [101] for related topics.)

As mentioned in the beginning, we hope that our weak relations would lead to more impressive formulas. Moreover, it would be desirable to understand such formulas in a geometric way, as in the case of Göttsche's formula and the Nakajima-Grojnowski construction.

The study of wall-crossing phenomena for Donaldson-Thomas invariants has been developing intensively. (See [4], [73], [114], [123], for example.) The author expects that we will have wall crossing phenomena in many other situations. It would be very interesting if some of the techniques used in this monograph could be applied.

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I started this project in 1998, my last year at the graduate school in Kyoto University. I continued the study in Osaka City University and the Institute for Advanced Study. I resumed it in 2006 at the Max-Planck Institute for Mathematics, and continued the study at the Institut des Hautes Études Scientifiques. Then, the final version was written at Kyoto University, Department of Mathematics, and Research Institute for Mathematical Sciences. During the course of the study, I visited the Tata Institute of Fundamental Research and the International Centre for Theoretical Physics. I am heartily grateful to the institutions and the colleagues for the excellent mathematical environment, and I would like to express my deep gratitude to those who kindly supported me. The final manuscript is completely different from the first version after several major revisions, and it took very long time for publication partially because of my poor ability in writing, which I apologize to those who kindly showed their interest in this study. Anyway, I feel relief that the manuscript is published at last.

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Kyoto, June 2008 *Takuro Mochizuki*

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