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P.S. Landweber (Ed.)

Elliptic Curves and Modular Forms in Algebraic Topology

Proceedings, Princeton 1986

Springer-Verlag

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Proceedings of a Conference held at
the Institute for Advanced Study
Princeton, Sept. 15–17, 1986

Springer-Verlag
Berlin Heidelberg New York London Paris Tokyo

Editor

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Mathematics Subject Classification (1980): 11F11, 33A25, 33A45, 55N22, 57S15, 81E99

ISBN 3-540-19490-8 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-19490-8 Springer-Verlag New York Berlin Heidelberg

Printed in Germany

Printing and binding: Druckhaus Beltz, Hemsbach/Bergstr.
2146/3140-543210

Preface

This volume contains the proceedings of a conference held September 15-17, 1986 at the Institute for Advanced Study in Princeton, New Jersey.

The introductory article provides an account of the recent history of the field of elliptic genera and elliptic cohomology, the central theme of the conference.

The main surprise at the conference was that its original conception was too narrow, and that geometry and physics also enter prominently into this area. For this, see the paper by Ed Witten.

I am grateful to Noriko Yui for permitting her paper on the formal groups of Jacobi quartics, an especially relevant topic for the study of elliptic genera, to be included in this volume.

Thanks are due to David and Gregory Chudnovsky for the suggestion to hold such a conference, and to Bob Stong for substantial advice throughout. It is also a pleasure to thank the School of Mathematics at the Institute for Advanced study, for providing the setting for the conference, and especially Linda Sheldon for much aid. Partial financial support was provided by the National Science Foundation.

Conference Talks

S. Ochanine, Elliptic genera for S^1 manifolds

P. Landweber, Periodic cohomology theories defined by elliptic curves

D. Chudnovsky and G. Chudnovsky, Elliptic formal groups over $\mathbb{Z}$ and $\mathbb{F}_p$ in applications to topology, number theory and computer science

R. Stong, Dirichlet series and homology theories

D. Ravenel, BP-theory for number theorists

M. Hopkins, Characters and generalized cohomology

J. Morava, The Weil group as automorphisms of the extraordinary K-theories

D. Zagier, Modular forms, elliptic functions, Jacobi forms

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ELLIPTIC GENERA: AN INTRODUCTORY OVERVIEW

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The aim of this article is to give an account of the development of the field of elliptic genera. Much of the interest since the fall of 1986 has concerned geometry and physics, but the emphasis here will be on the growth of this area between the fall of 1983 and the fall of 1986, for the benefit of those attracted to the field and wanting to know something of its origins.

At the outset, let me point out that my work in this are has been joint with Bob Stong. I would also like to thank Don Zagier for suggesting the usefulness of such an introductory article.

<u>Prehistory</u>. Here are some of the older results that have played major roles in recent developments.

A. Let the circle group S^1 act smoothly and nontrivially on a closed connected spin manifold M^{2n}. Then one has the Dirac operator [4]

$$D:\Gamma(S^+) \longrightarrow \Gamma(S^-)$$

on spinor fields, for which

$$\text{index}(D) = \hat{A}(M^{2n})$$

where

$$\hat{A}(M) = \hat{\mathcal{A}}(M)[M], \ \hat{\mathcal{A}}(M) = \prod_i \frac{u_i/2}{\sinh(u_i/2)}$$

if M has total Pontrjagin class

$$p(M) = \prod_i (1+u_i^2).$$

Atiyah and Hirzebruch showed in 1970 [3] that the existence of such an S^1-action implies that $\hat{A}(M) = 0$. Indeed, in case the S^1-action lifts to the principal spin bundle defining the spin structure, so that one has a refined equivariant index

$$\text{index}^{S^1}(D) \in R(S^1),$$

they proved that this character-valued index vanishes.

B. $\Omega_*^U(X)$ denotes the complex bordism of a space X, a module over the complex bordism ring Ω_*^U. One has the <u>Todd genus</u>

$$Td:\Omega_*^U \longrightarrow \mathbb{Z},$$

namely,

$$Td(M) = \prod_i \frac{u_i}{1-e^{-u_i}} [M]$$

if the U-manifold (with a complex structure on its stable tangent bundle) M has total Chern class

$$c(M) = \prod_i (1+u_i).$$

Conner and Floyd [10] showed in 1966 that the homology theory

$$K_*(X) = K_0(X) \oplus K_1(X)$$

dual to complex K-theory can be obtained from complex bordism by tensoring:

$$\Omega_*^U(X) \otimes_{\Omega_*^U} \mathbb{Z}_{Td} \cong K_*(X),$$

where we view

$$\Omega_*^U(X) = \Omega_{even}^U(X) \oplus \Omega_{odd}^U(X)$$

and write $\mathbb{Z}_{Td}$ to indicate that $\mathbb{Z}$ is made an algebra over Ω_*^U via the Todd genus.

C. Let $R(t) = 1-2\delta t^2 + \epsilon t^4$ with δ, ϵ complex numbers, and suppose that

$$\int_0^x \frac{dt}{\sqrt{R(t)}} + \int_0^y \frac{dt}{\sqrt{R(t)}} = \int_0^{F(x,y)} \frac{dt}{\sqrt{R(t)}} .$$

Thus $F(x,y)$ expresses the addition formula for such an elliptic integral. In 1756, Euler [11] gave the formula

$$F(x,y) = \frac{x\sqrt{R(y)} + y\sqrt{R(x)}}{1-\epsilon x^2y^2} .$$

This formal group is quite beautiful, and is central to the number-theoretic study of elliptic genera.

Turning to more recent times we begin with a problem raised by Ed Witten [26] in October 1983.

0. Again, let S^1 act smoothly on a closed spin manifold M^{2n}, but now consider the twisted Dirac operator

$$D_T:\Gamma(S^+ \otimes T) \longrightarrow \Gamma(S^- \otimes T)$$

for spinor fields with coefficients in the tangent bundle T. Then

$$\text{index } D_T = \hat{A}(M)ch(T)[M],$$

where $ch(T)$ denotes the Chern character of the complexification of T. Assume now that the S^1-action lifts to the principal spin bundle defining the spin structure on M. Then one has a character-valued index

$$\text{index}^{S^1}(D_T) \in R(S^1),$$

and Witten [26, §V] asked if this was in fact constant (as a representation, or character). He found this to hold for actions on homogeneous spaces, and suggested that one might apply bordism techniques to prove it in general.

1. With this question in mind, Lucilia Borsari [6,7] took up the problem of analyzing the bordism of circle actions on spin manifolds. She examined a simplified problem, by dealing with semifree S^1-actions on spin manifolds (actions free on the complement of the fixed point set) and tensoring the bordism groups with the rationals. The most interesting problem raised by this study was to determine the ideal

$$I_* \subset \Omega_*^{SO} \otimes \mathbb{Q}$$

(note that $\Omega_*^{Spin} \otimes \mathbb{Z}[\frac{1}{2}] \longrightarrow \Omega_*^{SO} \otimes \mathbb{Z}[\frac{1}{2}]$ is an isomorphism) generated by spin manifolds admitting semifree S^1-actions of odd type (i.e., the action on each component does not lift to the spin bundle). One sees that I_* is also generated as an ideal by all bordism classes $[\mathbb{C}P(V^{2m})]$, $V^{2m} \longrightarrow B$ being a complex vector bundle of even complex dimension over an oriented base. One sees, further, that both the signature [8] and $\hat{A}$-genus [5] vanish on all $\mathbb{C}P(V^{2m})$'s, a promising prospect.

2. A closer look at Borel and Hirzebruch's work from 1958-59 [5] revealed that on the bundles $\mathbb{C}P(V^{2m})$, with fibres the homogeneous spaces

$$\mathbb{C}P^{2m-1} = U(2m) \ / \ U(1) \times U(2m-1),$$

not only $\hat{A}$ but also (using an evident shorthand)

$$\hat{A}\ \mathrm{ch}T, \hat{A}\ \mathrm{ch}(\Lambda^2 T),$$
$$\hat{A}\ \mathrm{ch}(\Lambda^3 T + T \otimes T)$$

and several more characteristic numbers of the form

$$\hat{A}\ \mathrm{ch}(\Lambda^k T + \text{lower terms})$$

vanish [15]. We wanted to understand what was behind this.

3. As to the ideal I_*, one has

$$\Omega_*^{SO} \otimes \mathbb{Q} = \mathbb{Q}[x_4, x_8, x_{12}, x_{16}, \ldots]$$

and can choose

$$x_4 = [\mathbb{C}P^2], \quad x_8 = [\mathbb{H}P^2]$$

and $x_{12}, x_{16}, \ldots \in I_*$. We conjectured that

$$I_* = (x_{12}, x_{16}, \ldots),$$

which was supported by the results given above.

Serge Ochanine [16] proved this equality by introducing the notion of an <u>elliptic genus</u>

$$\varphi : \Omega_*^{SO} \longrightarrow R.$$

This means a ring homomorphism (a multiplicative genus) to a commutative $\mathbb{Q}$-algebra, with $\varphi(1) = 1$, so that its <u>logarithm</u>

$$g(x) = \sum_{n \geq 0} \frac{\varphi(\mathbb{C}P^{2n})}{2n+1} x^{2n+1}$$

is an elliptic integral

$$g(x) = \int_0^x \frac{dt}{\sqrt{R(t)}}$$

with

$$R(t) = 1 - 2\delta t^2 + \epsilon t^4 \quad (\delta, \epsilon \in R).$$

He used residues of elliptic functions to prove that, for φ an elliptic genus, one has

$$\varphi(\mathbb{C}P(V^{2m})) = 0,$$

Since $\varphi(\mathbb{C}P^2) = \delta$ and $\varphi(\mathbb{H}P^2) = \epsilon$ for an elliptic genus, it follows easily that I_* can also be characterized as the elements in $\Omega_*^{SO} \otimes \mathbb{Q}$ killed by all elliptic genera.

4. Returning to item 2, we showed that there is an elliptic genus

$$\rho:\Omega_*^{SO} \longrightarrow \mathbb{Q}[[q]],$$
$$\rho(M) = \sum_{k\geq 0} \rho_k(M)q^k,$$

with the coefficients in this power series of the form given there:

$$\rho_k(M) = \hat{A}(M)\mathrm{ch}\rho_k(T)[M]$$

for suitable virtual bundles

$$\rho_k(T) = (-1)^k\Lambda^k(T) + \text{lower terms}$$

depending on the tangent bundle [15]. Strictly, we saw that $\rho_k(T) \in KO(M)$ for low k, but at first only knew that $\rho_k(T) \in KO(M) \otimes \mathbb{Q}$ in general.

5. We needed number theorists to clarify the situation. We learned from David and Gregory Chudnovsky [9] and Don Zagier [29] that ρ maps to <u>modular forms</u> (see below) for

$$\Gamma_0(2) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL_2\mathbb{Z} \mid c \text{ even}\right\},$$

$\rho(M^{4n})$ being the q-expansion at ∞ of a modular form of <u>weight</u> 2n. We learned explicit expressions for δ and ϵ, as modular forms of weights 2 and 4, respectively, and that $\rho_k(T) \in KO(M)$ for all k. But we still did not know what was behind this.

To say that f is a modular form for $\Gamma_0(2)$ of weight $2n$ means that $f:H \longrightarrow \mathbb{C}$ is a holomorphic function on the upper half-plane, for which

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{2n}f(\tau)$$

for $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma_0(2)$, and with

$$f(\tau) = \sum_{k\geq 0} a_k q^k \qquad (q = e^{2\pi i\tau})$$

and a similar holomorphicity condition at the other cusp $\tau = 0$. It is customary to identify a modular form with its q-expansion at ∞.

6. One sees easily that an elliptic genus always sends Ω_*^{SO} into $\mathbb{Z}[\frac{1}{2}][\delta,\epsilon]$; we shall now take δ and ϵ to be indeterminates of dimensions 4 and 8, respectively, and view $\mathbb{Z}[\frac{1}{2}][\delta,\epsilon]$ as an algebra

over Ω_*^{SO} via the corresponding elliptic genus. We asked if there was an underlying homology theory satisfying a "Conner-Floyd theorem" (see item B). Working with Doug Ravenel [13,14], we sought a homology theory with the homology of a point being

$$\mathbb{Z}[\tfrac{1}{2}][\delta,\epsilon,\Delta^{-1}],$$

where $\Delta = \epsilon(\delta^2-\epsilon)^2$ (the discriminant). Indeed, we proved that

$$X \longrightarrow \Omega_*^{SO}(X) \otimes_{\Omega_*^{SO}} \mathbb{Z}[\tfrac{1}{2}][\delta,\epsilon,\Delta^{-1}]$$

is a homology theory, with periodicity of dimension 24. We could also invert merely ϵ or $\delta^2-\epsilon$, and so get homology theories with periodicity of dimension 8.

Thus, we saw the existence of homology and cohomology theories, related to elliptic curves in the Jacobi quartic form

$$y^2 = 1-2\delta x^2 + \epsilon x^4$$

and to modular forms. These are complex-oriented theories, for which the corresponding formal group is the one found by Euler (see item C). At this point a large number of questions suggested themselves.

7. In September 1986, Ed Witten [27] shed considerable light on the "universal" elliptic genus

$$\rho:\Omega_*^{SO} \longrightarrow \mathbb{Q}[[q]]$$

mentioned above. His prescription, with origins in quantum field theory, is as follows. For an oriented manifold M with tangent bundle T, build

$$S_q(T) = \sum_{n\geq 0} S^n(T)q^n, \quad \Lambda_q(T) = \sum_{n\geq 0} \Lambda^n(T)q^n$$

from the symmetric and exterior powers of T, and then write

$$\begin{aligned} R(T) &= \bigotimes_{\substack{\ell>0 \\ \ell \text{ even}}} S_{q^\ell}(T) \otimes \bigotimes_{\substack{\ell>0 \\ \ell \text{ odd}}} \Lambda_{-q^\ell}(T) \\ &= \sum_{k\geq 0} R_k(T)q^k. \end{aligned}$$

One finds that

$$R_0 = 1$$
$$R_1 = -T$$

$$R_2 = \Lambda^2 T + T$$
$$R_3 = -(\Lambda^3 T + T \otimes T)$$

and that

$$(-1)^k R_k(T) = \Lambda^k T + \text{lower terms}.$$

Then it is easily verified [13,29] that our previous $\rho(M) \in \mathbf{Q}[[q]]$ coincides with

$$\hat{A}(M)\ \mathrm{ch}\left\{\frac{R(T)}{R(1)^{\dim M}}\right\}[M].$$

8. Witten also returned to his original question, about the constancy of the character-valued index of D_T, for an S^1-action on a spin manifold. But now we have all the $R_k(T)$ and $\rho_k(T)$ for $k \geq 0$, and so can ask that

$$\mathrm{index}^{S^1}(D_{\rho_k T}) \in R(S^1)$$

be constant for all $k \geq 0$. I.e., one wants the entire equivariant elliptic genus to be constant.

Serge Ochanine had previously studied the question in this form [17,18] and came close to solving it. As a formality, if a compact Lie group G acts smoothly on a closed oriented manifold M, and if $\varphi: \Omega_*^{SO} \longrightarrow R$ is a multiplicative genus for oriented manifolds, one can define an equivariant extension [17,18]

$$\varphi^G(M) \in \prod_{n \geq 0} H^n(BG; R).$$

One then wants to prove that

$$\varphi^G(M) \in H^0(BG; R),$$

provided that G is connected, M is a spin manifold, and φ is an elliptic genus. It suffices to deal with the case $G = S^1$, which Ochanine did provided the action is semifree [17] or preserves a weakly complex structure [18].

Witten gave arguments in [27] and [28] suggesting that the constancy could be proved in the case of general S^1-actions on spin manifolds. Since his arguments rely on unproved properties of the super-symmetric nonlinear sigma model (or equivalently, of Dirac-like operators on the free loop space $\mathcal{L}M$), the task remained to make his ideas into a rigorous proof.

9. Another very pleasant surprise was to learn that other physicists had independently developed some of these ideas. Although I do not feel competent to review these developments, I do want to cite work by K. Pilch, A. Schellekens and N. Warner ([19,20,21,22]) on anomalies and string theory, leading to modular forms for $SL_2(\mathbb{Z})$ in the case of spin manifolds with $p_1 = 0$; and also by O. Alvarez, T. Killingback, M. Mangano and P. Windey ([1,2,12,25]) on loop space index theorems, in which they give a detailed analysis of the index of the Dirac-Ramond operator, the main theme being to extend to loop spaces the path integral proof of the Atiyah-Singer index theorem.

10. The constancy of equivariant elliptic genera for S^1-actions on spin manifolds has now been proved by Cliff Taubes [24]. He makes Witten's program rigorous. One begins with a spin manifold and views

$$M \subset \mathcal{L}M,$$

$\mathcal{L}M$ being the free loop space and M the constant loops. One wants to generalize the Dirac operator on M to a Dirac-like operator on $\mathcal{L}M$. Taubes finds it sufficient to deal with the normal bundle of M in $\mathcal{L}M$. That is, he deals with "small loops" in M centered above points $p \in M$; a small loop is a map $S^1 \longrightarrow TM_p$ whose Fourier expansion has zero constant term. An important feature is the evident "internal" S^1-action on $\mathcal{L}M$ (in addition to the "geometric" S^1-action arising from an action of S^1 on M), having M as its fixed point set. The rather difficult argument given by Taubes [24] has been simplified in further work of Raoul Bott and Taubes.

11. Prospects. Just as index theory for elliptic operators leads to (and requires) K-theory, one can fairly expect that index theory on $\mathcal{L}M$ will lead to elliptic cohomology. At the moment, the main problems are to give a geometric description of elliptic cohomology, and to clarify the connection with index theory on free loop spaces. One expects a prominent role for loop groups and their representations, and further relations with quantum field theory. One also wants to put elliptic cohomology to further use in the realm of topology.

A superb and timely review of this whole field has been given recently by Graeme Segal [23].

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ELLIPTIC FORMAL GROUPS OVER $\mathbb{Z}$ AND $\mathbb{F}_p$ IN APPLICATIONS TO NUMBER THEORY, COMPUTER SCIENCE AND TOPOLOGY.

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Introduction.

Formal groups have long been used to solve problems in algebraic geometry, algebraic number theory and topology. In this paper, we describe a few more applications of many concepts borrowed from formal groups. One of them is the relationship between the integrality conditions on power series expansions of functions (representing, say, logarithms of formal groups) and the algebraic and analytic properties of these functions. Among the formal groups that we consider those associated with algebraic curves, particularly with elliptic curves, and, in general, with laws of addition on Abelian varieties, play the most important role. In addition to topology our main areas of focus in this paper are: number-theoretic properties of differential equations (the Grothendieck problem), the uniformization problem and integrality conditions (the Tate conjecture), congruences for coefficients of algebraic differentials (related to Schur congruences), and a variety of applications of interpolation on algebraic curves over finite fields.

This work started a few years ago when we got interested in formal groups in connection with the Grothendieck conjecture (see §1). Our interest in formal groups and their application grew as we met Peter Landweber and got involved by him and Bob Stong in a variety of exciting problems centered around formal groups, characteristic classes and modular forms (see §7). For the last two years, Landweber and Stong conducted an international seminar by correspondence, open to everybody, that generated considerable progress in this field, and this conference is just a physical realization of this virtual seminar. In this paper, which follows our lecture, we will deviate from algebraic topology into computer problems, still firmly holding onto elliptic curves.