# Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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# PREFACE

These are the proceedings of the colloquium on potential theory held in Copenhagen, May 14th - 18th, 1979, on the occasion of the 500th anniversary of the University of Copenhagen. The colloquium was sponsored by the Danish-French Society for Scientific Research and Exchange, and took place at the H. C. Ørsted Institute. There were 71 participants from 18 countries. The scientific programme comprised 44 lectures and a "Table ronde", where open problems were discussed (cf. the problem section at the end of these proceedings).

Potential theory has developed in several directions and has interfaces with a diversity of branches of pure and applied mathematics. It has been the particular aim and hope of the organisers that this colloquium should contribute to maintaining and promoting contact and cooperation between potentialists working with different aspects of potential theory. We take the opportunity of thanking all the participants for their presence and their scientific contribution to the Colloquium.

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Organisers and editors

The following lectures are not published in these proceedings.

BAUER, H.: Dirichlet problem for the Choquet boundary and Korovkin closure.

BENEDICKS, M.: Positive harmonic functions vanishing on the boundary of some domains in $R^n$ .

BOUKRICHA, A.: The Poisson space ${}^{c}P_x$ for $\Delta u = cu$ with rotation free $c$ .

BOULEAU, N.: Processus associé à un espace biharmonique et théorie du potentiel pour certains systèmes différentiels.

DENY, J.: Sur quelques applications de la TV-inégalité, d'après des résultats inédits de G. Choquet.

ÉLIE, L.: Fonctions harmoniques positives sur le groupe affine.

ESSÉN, M.: On the covering properties of certain exceptional sets. (Joint work with H. L. JACKSON)

FUGLEDE, B.: Invariant characterization of the two topologies on a harmonic space.

GAUTHIER, P. M.: Approximation analytique.

HANSSON, K.: Strong capacitary inequalities and imbedding theorems of Sobolev type.

HAYMAN, W. K.: Local approximation to plane harmonic functions by functions of restricted growth.

HYVÖNEN, J.: On the harmonic continuation in harmonic spaces.

KANDA, M.: A measure which determines semipolar sets.

LAINE, I.: Full-hyperharmonic structures on harmonic spaces.

LYONS, T. J.: Function algebras and finely holomorphic functions.

MOKOBODZKI, G.: Potentiels semi-réguliers et dualité.

PIERRE, M.: Potentiels paraboliques et équations d'évolution avec obstacles.

RAMASWAMY, S.: Fine connectedness and the minimum principle for excessive functions.

SCHIRMEIER, U.: Convergence properties for harmonic spaces in duality.

SJÖGREN, P.: On the eigenfunctions of the Laplacian in a symmetric space.

STOICA, L.: The addition of local operators on product spaces.

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## CONTENTS

# ADMISSIBLE SUPERHARMONIC FUNCTIONS

V. Anandam

## I. Introduction

In the study of potential theory, two cones of functions (the class of positive superharmonic functions and the class of positive potentials) play an important rôle. But these two classes of functions are not very significant when there does not exist any positive potential in the whole space as in $R^2$. Instead, in this case, we have two other cones of functions which are quite interesting.

We place ourselves in the axiomatic case of M.Brelot without any positive potential. The two cones of functions are the class of admissible superharmonic functions (that is, those superharmonic functions which have harmonic minorants outside a compact set) and the class of pseudo-potentials. A superharmonic function is admissible if and only if it is the sum of a pseudo-potential and a harmonic function.

We give here some important properties of pseudo-potentials and characterise them by means of their balayage functions. Using these properties, a notion of capacity is defined, which appears as a slight variant of the logarithmic capacity in the classical case. Some of these results have appeared elsewhere in a scattered form (see [2] and [3] ).

In the special case of $R^2$, an admissible superharmonic function is simply a superharmonic function whose total associated measure in a local Riesz representation is finite; and a superharmonic function u is a pseudo-potential if and only if -u is a subharmonic function of potential type as defined by Prof. M. Arsove (that is, the total associated measure of -u is finite and the order of -u is 0). More details in this direction, particularly in the context of integral representation of a class of superharmonic functions including the pseudo-potentials by means of a generalised logarithmic kernel, will be given in a paper written in collaboration with Prof. M. Brelot.

## II. Admissible superharmonic functions in a B.S. space.

Let $\Omega$ be a B.S. space; that is, a harmonic space satisfying the axioms of M. Brelot, with constants harmonic and having no potential $> 0$ in the whole space. Fix an ultrafilter e finer than the filter of sections of the set of relatively compact open sets $\omega$ of $\Omega$ . If $\bar{H}_u^\omega$

stands for the upper Dirichlet solution, let $D(u) = \lim_{e} \bar{H}_u^{\omega}$. Also we fix an outer regular compact set k and a non constant harmonic function $H \geq 0$ in $\Omega - k$ tending to 0 on $\partial k$.

A superharmonic function u in $\Omega$ is <u>admissible</u> if it has a harmonic minorant outside a compact set. Thus u is admissible if and only if its flux at infinity is finite (for the notion of flux, see [1]).

## 1. Pseudo-potentials.

A superharmonic function u is a <u>B.S. potential</u> if for some ß, $D(u-\beta H) = 0$ which implies that $\lim_{x \to \alpha} \inf u(x) - \beta H(x) > -\infty$ where $\alpha$ stands for the point at infinity and consequently u is an admissible superharmonic function with flux at infinity ß. We shall denote by $L^*$ the class of admissible superharmonic functions which are B.S.potentials up to additive constants.

An admissible superharmonic function u with flux ß is a <u>pseudo-potential</u> if $h-\beta H$ is bounded outside a compact set where h is the greatest harmonic minorant of u outside a compact set.

<u>Proposition 1</u>: Any admissible superharmonic function u can be written as the sum of a pseudo-potential and a harmonic function.

<u>Proof</u>: If h is the greatest harmonic minorant of u outside a compact set, then $h = \beta H$ + a harmonic function v in $\Omega$ + a bounded harmonic function outside a compact. This remark is sufficient to prove the proposition.

Recall that two admissible superharmonic functions are <u>equivalent</u> if the difference between their greatest harmonic minorants outside a compact set is bounded. We remark in passing that two equivalent functions have the same flux; and two admissible superharmonic functions u and v with the same flux are equivalent if $u \leq v$.

<u>Proposition 2</u>: An admissible superharmonic function equivalent to a pseudo-potential is a pseudo-potential.

<u>Proof</u>: This proposition follows from the fact that an admissible superharmonic function is a pseudo-potential if and only if its greatest harmonic minorant outside a compact set is of the form $\beta H$ + a bounded harmonic function.

<u>Corollary 1</u>: If u and v are pseudo-potentials, then inf(u,v) also is a pseudo-potential.

<u>Proof</u>: If flux u $\leq$ flux v, we obtain immediately that inf(u,v) is

equivalent to u and hence the result.

Corollary 2: An admissible superharmonic function u is a pseudo-potential if and only if u is equivalent to a B.S. potential.

Proof: It is enough to remark that if $\beta$ is the flux u at infinity, $\beta H$ extended by 0 on k is a B.S. potential and that a B.S. potential is necessarily a pseudo-potential.

We recall that $\Omega$ is said to be of harmonic dimension 1 if every positive harmonic function outside a compact set is of the form $\beta H$ + a bounded function.

Theorem 3: In a B.S. space $\Omega$ the following are equivalent:

i) $\Omega$ is of harmonic dimension 1.

ii) If s is a superharmonic function majorizing a pseudo-potential, then s is a pseudo-potential.

iii) If s is an admissible superharmonic function majorized by a pseudo-potential, then s is a pseudo-potential.

iv) Any upper bounded admissible superharmonic function is a pseudo-potential.

Proof: To prove this theorem, we use proposition 1 and the fact that if $\Omega$ is of harmonic dimension > 1, then there exists a non constant harmonic function h in $\Omega$ such that $h \leq H$ outside a compact set.

Corollary 1: In a B.S. harmonic space of dimension 1, a superharmonic function u is a pseudo-potential if and only if u majorizes some $v \in \overset{*}{L}$.

Corollary 2: In a B.S. harmonic space of dimension 1, a superharmonic function u is a pseudo-potential if and only if inf(u,0) is a pseudo-potential.

Proof: If u is a pseudo-potential, inf(u,0) is admissible and hence a pseudo-potential by iv). On the other hand, if inf(u,0) is a pseudo-potential, then u is a pseudo-potential by ii).

## 2. Balayage.

In [1] a method was given to obtain the balayage of an admissible superharmonic function on a nonpolar compact set. That is, if v is an admissible superharmonic function and e is a nonpolar compact set in $\Omega$, then one can define an admissible superharmonic function u with the properties: $u \leq v$ in $\Omega$, $u = v$ in $\mathring{e}$, u is harmonic in $\Omega - e$ and flux u = flux v. It is clear that this method is meaningful only for the class of admissible superharmonic functions.

Later Guillerme [6] generalised this method to consider balayage of an admissible superharmonic function on any nonpolar set. One supposes here that $\Omega$ has a countable base.

Let u be an admissible superharmonic function and e be a set not locally polar. Let B(u,e) stand for the family of superharmonic functions majorizing u on e and outside a compact set; and F(u,e) the family of superharmonic functions which majorize u on e and are equivalent to u at infinity. Then $\inf \{v : v \in B(u,e)\} = \inf \{v : v \in F(u,e)\}$ and the common value $B_u^e$ is such that $B_u^e \leq u$ in $\Omega$ , $B_u^e = u$ on e, $B_u^e$ is harmonic in $\Omega - \bar{e}$ and $\hat{B}_u^e$ is superharmonic equivalent to u at infinity.

It is shown further that $B_u^e$ has many useful properties of réduite; among them are:

i) if e is fine open, $B_u^e = \sup \{ B_u^k : k \text{ compact} \subset e\}$,

ii) if u is finite continuous, $B_u^e = \inf \{ B_u^\omega : \omega \text{ open} \supset e\}$,

iii) $B_u^{C \cup D} + B_u^{C \cap D} \leq B_u^C + B_u^D$,

iv) if $e = \bigcup e_n$ where $e_n$ is an increasing sequence of nonpolar sets and if u is finite then $B_u^e = \lim B_u^{e_n}$.

Further we remark that if q is a pseudo-potential, then $\hat{B}_q^e$ is a pseudo-potential; more over, if e is compact $\hat{B}_q^e \in L^*$.

<u>Theorem 4</u>: Let v be an admissible superharmonic function in $\Omega$ with countable base. If v is a pseudo-potential then $\hat{B}_v^{\{v \leq 0\}} = \inf(v,0)$ in $\Omega$. The converse also is true if $\Omega$ is of harmonic dimension 1.

<u>Proof</u>: Since two pseudo-potentials with the same flux at infinity are equivalent, inf(v,0) is equivalent to v. Hence $\hat{B}_v^{\{v \leq 0\}} \leq \inf(v,0)$.

On the other hand, if s is a superharmonic function majorizing v on $\{v \leq 0\}$ and the complement of a compact set then $s \geq 0$ in $\{v > 0\}$ and consequently $s \geq \inf(v,0)$ in $\Omega$. Hence $\hat{B}_v^{\{v \leq 0\}} \geq \inf(v,0)$.

To prove the second part, write v = p+h as the sum of a pseudo-potential and a harmonic function. Then the greatest harmonic minorant u of v, and hence that of the equivalent function inf(v,0), outside a compact set is of the form $\beta H$ + h + a bounded harmonic function. Since u is bounded above and $\Omega$ is of harmonic dimension 1, h should be a constant. This completes the proof of the theorem.

<u>Theorem 5</u>: Any pseudo-potential u is the supremum of an increasing sequence of finite continuous $L^*$-potentials with compact support.

<u>Proof</u>: Using balayage, with usual arguments, one shows that (corollary 5.6 [6] ) $u = \sup u_n$ where $u_n$ is an increasing sequence of finite conti-

nuous superharmonic functions with compact (harmonic) support and equivalent to u. Since $u_n$ is equivalent to the pseudo-potential u, $u_n$ is a pseudo-potential; since it has compact support also, $u_n \in \overset{*}{L}$.

## 3. Capacity.

In this paragraph also $\Omega$ is assumed to have a countable base.

We introduce a set function c defined on the class of compact sets contained in a fixed relatively compact domain X of $\Omega$. Making use of certain properties of admissible superharmonic functions, we show that this set function is increasing and right-continuous; it is also strongly subadditive in the sense that if $e_1$ and $e_2$ are two compact sets in X, $c(e_1 \cup e_2)+c(e_1 \cap e_2) \leq c(e_1)+c(e_2)$.

We define then, in the usual manner, the following two set functions on the subsets of X:

$$c_*(e) = \sup \{ c(k) : \text{ compact } k \subset e \} \quad \text{and}$$
$$c^*(e) = \inf \{ c_*(\omega) : \text{ open } \omega \supset e \}.$$

It turns out that $c^*$ is a true capacity defined on all subsets of X. This true capacity appears as a slight variant of the logarithmic capacity in the classical case.

To define c, let us take some outer regular compact set K such that $\bar{X} \subset \overset{\circ}{K}$. Let q be the unique superharmonic function in $\Omega$ such that q = 0 on K, harmonic in $\Omega$ - K, tending to 0 on $\partial K$ and q+H is bounded outside a compact set.

Definition 6: The set function c on the compact sets e of X are defined as follows: if e is nonpolar $c(e) = D(B_q^e+H)$; and if e is locally polar $c(e) = -\infty$.

Remarks: 1) The value c(e) does not depend on the choice of X nor of K. 2) If the axiom D is satisfied locally in $\Omega$, we can prove that if e is a nonpolar compact set, then there exists a unique pseudo-potential p with flux at infinity -1 such that p is harmonic outside e, $p \leq -c(e)$ in $\Omega$ and $p = -c(e)$ q.e. on e.

Theorem 7: The set function c is increasing, right-continuous and strongly subadditive on the class of all compact sets of X. The extended function $c^*$ is a true capacity defined on all subsets of X.

Proof: The properties of c are the consequences of the corresponding properties of balayage recalled in the previous paragraph. The only point that needs a special consideration is to prove that c(e) is right-continuous when e is locally polar.

In this case note that we can choose a pseudo-potential p with compact support such that $p = \infty$ on e and $D(p+H) = 0$. Let $\omega$ be an open set in X such that $p > n$ on $\omega$. Then for any compact set $e_1$ such that $e \subset e_1 \subset \omega$, we have $B_q^{e_1} \leq p-n$ since q and p-n are equivalent at infinity and on $e_1$, $p-n \geq 0 = q$. Consequently $c(e_1) \leq D(p+H-n) = -n$.

To prove the second part, we use again the recalled properties of balayage and show that for any open set $\omega \subset X$, $c_*(\omega) = D(B_q^{\omega}+H)$ and for any arbitrary set e in X, $c^*(e) = D(B_q^e+H)$.

The usual arguments then show that $c^*$ is a true capacity; the fact that $c^*$ can take the value $-\infty$ does not really pose great difficulties. In fact, the only difficult condition to be verified is that if e is the union of an increasing sequence of sets $e_n$ in X then $c^*(e)=\lim c^*(e_n)$.

Now if each $e_n$ is locally polar, then $c^*(e) = -\infty = c^*(e_n)$. We shall suppose, therefore, that $e_n$ is not locally polar. Since $B_q^e = \sup B_q^{e_n}$ and $B_q^{e_n}$ are all harmonic in $\Omega - \bar{X}$, if $\omega$ is a domain containing $\bar{X}$, $B_q^e - B_q^{e_n} < \varepsilon$ on $\partial\omega$ if n is large. This implies that $B_q^e - B_q^{e_n} \leq \varepsilon$ outside $\omega$ since $B_q^e - B_q^{e_n}$ is bounded in the neighbourhood of the point at infinity. Consequently $c^*(e) = \lim c^*(e_n)$.

## III. Admissible superharmonic functions in $R^2$.

In this section we consider briefly some of the results relating to pseudo-potentials in the context of classical development in $R^2$. More details can be obtained in [4] .

First we note that a superharmonic function u in $R^2$ with associated measure $\mu$ is admissible if and only if $\|\mu\| = \int d\mu$ is finite.

To see this, we take an inversion with the origin as pole which transforms u into u' and $\mu$ into the measure $\mu'$ associated with u'. If G(x,y) is the Green function on some $B_0^r$, u'(x) has a harmonic minorant in $B_0^r - \{0\}$ if and only if u'(x) equals $\int G(x,y)\, d\mu'(y)$ in $B_0^r - \{0\}$ up to a harmonic function, or equivalently $\mu'(N)$, with $\mu'(0) = 0$, is finite for some neighbourhood N of 0.

Let us consider now a generalised logarithmic kernel $\Lambda(x,y)$ in $R^2$ defined as:

$$\Lambda(x,y) = \begin{cases} \log(1/|x-y|) & \text{if } |y| < 1 \\ \log(1/|x-y|) - \log(1/|y|) & \text{if } |y| \geq 1. \end{cases}$$

As to the existence of the associated $\Lambda$-potentials with respect to arbitrary measures $\mu \geq 0$, we have the following result:

<u>Theorem 8</u>: Let $\mu \geq 0$ be a Radon measure in $R^2$. Then the following are equivalent:

i) $\int \Lambda(x,y)\, d\mu(y)$ is defined everywhere and superharmonic in $R^2$.

ii) $\int_R^\infty d\mu(y)/|y|$ is finite for some R.

iii) $\int_R^\infty \mu(B_O^r)/r^2\, dr$ is finite for some R.

<u>Consequence</u>: Any admissible superharmonic function u with associated measure $\mu \geq 0$ has a unique representation $u(x) = \int \Lambda(x,y)\, d\mu(y) + h(x)$ as the sum of a $\Lambda$-potential and a harmonic function.

Recall that for an increasing positive function M(r), the order $\lambda$ of M(r) is defined to be $\lambda = \limsup (\log M(r)/\log r)$ and when $0 < \lambda < \infty$, M(r) is said to be of convergence class if $\int_R^\infty M(r)/r^{\lambda+1}\, dr$ is finite.

Let u be a superharmonic function with associated measure $\mu$. Then,

i) the order of u = the order of B(r,-u) where $B(r,-u) = \max_{x=r} -u(x)$

ii) the genus of u is the smallest positive integer g for which $\int_1^\infty \mu(B_O^r)/r^{g+2}\, dr$ is finite, and

iii) the Nevanlinna function of u is $N(r) = \int_1^r \mu(B_O^x)/x\, dx$.

<u>Theorem 9</u>: Let u be a superharmonic function in $R^2$. Then u is a $\Lambda$-potential up to a harmonic function, equivalently of genus 0, if and only if N(r) has order $< 1$ or of convergence class of order 1.

<u>Proof</u>: The condition on N(r) is the same as that $\int_R^\infty N(r)/r^2\, dr$ is finite or equivalently that $\int_R^\infty \mu(B_O^r)/r^2\, dr$ is finite. The proof is now completed using theorem 8.

<u>Lemma 10</u>: Let u be an admissible $\Lambda$-potential with associated measure $\mu$. Then $u(x) \geq -\|\mu\| \log(1+|x|)$.

This is immediate if we note that $\Lambda(x,y) \geq -\log(1+|x|)$ for all x and y. For when x is fixed,

if $|y| < 1$, $\log(1/|x-y|) \geq -\log(|x|+1)$ and

if $|y| \geq 1$, $\log(|y|/|x-y|) \geq -\log(1+(|x|/|y|)) \geq -\log(1+|x|)$.

<u>Theorem 11</u>: Let u be an admissible superharmonic function in $R^2$. Then the following are equivalent:

i) u is a $\Lambda^*$-potential (that is a $\Lambda$-potential up to a constant).

ii) outside a disc, u majorizes a harmonic function of the form $\beta \log|x|$ + a constant.

iii) $\liminf_{|x|\to\infty} (u(x)/\log|x|)$ is finite.

iv) order of u is 0.

Proof: As a consequence of lemma 10, it is immediate that i) $\Rightarrow$ ii) and iv); ii) implies obviously iii) which in turn implies iv).

It remains to see that iv) $\Rightarrow$ i). Since u is admissible, u is the sum of a $\wedge$-potential v and a harmonic function h. Now again by lemma 10, the order of v is 0; consequently, the order of h is 0 which shows that h is a constant.

Remarks: 1) As a consequence of theorem 16 of M. Arsove [5] and the above theorem we obtain immediately that a subharmonic function u is of potential type if and only if -u is an admissible $\wedge^*$-potential.

2) As a particular case of the axiomatic theory, let us say that an admissible superharmonic function u in $R^2$, with h as its greatest harmonic minorant outside a disc, is a pseudo-potential if for some $\beta$, $h(x)-\beta \log|x|$ is bounded outside a disc. As a consequence of the above theorem, we see that u is a pseudo-potential if and only if u is an admissible $\wedge^*$-potential (note that the property ii) implies that the greatest harmonic minorant of an admissible $\wedge^*$-potential outside a disc is of the form $\beta \log|x|$ + a bounded harmonic function). Further in this case, a B.S. potential is the same as a logarithmic potential.

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# PRINCIPE DE HARNACK A LA FRONTIERE ET PROBLEMES DE FRONTIERE DE MARTIN

par Alano ANCONA

Au cours de l'exposé au colloque, on a illustré par divers exemples une méthode conduisant à des estimations, au voisinage de points frontières, pour le quotient de certaines fonctions harmoniques sur un domaine; ce type d'estimation permet d'obtenir des propriétés géométriques de la compactification de Martin: citons d'abord l'extention d'un théorème de Hunt et Wheeden ([ 8] ,[ 9] ) aux opérateurs uniformément elliptiques à coefficients höldériens; si $L$ est un tel opérateur, $\Omega$ un domaine lipschitzien borné et greenien pour $L$, le $L$-compactifié de Martin de $\Omega$ est homéomorphe à $\overline{\Omega}$ , chaque point $P \in \partial\Omega$ étant associé à une minimale unique $K_p$, normalisée en un point fixé $O$, et $K_p$ tend vers zéro en tout point frontière $P' \neq P$; de plus, on peut étendre à ce cadre le théorème de Fatou sur l'existence de limites non-tangentielles de quotients de fonctions harmoniques ([ 1] ). Dans le cas du Laplacien, le principe de Harnack à la frontière à été obtenu par J.M.Wu ([ 12] ) par une méthode différente: B.Dahlberg a également donné dans ce cas une version de ce principe, mais la constante de son estimation dépend à priori du voisinage du point frontière considéré, et n'est pas uniforme par rapport à la constante de Lipschitz de $\Omega$ ([ 6] ).

Une autre application de la méthode est la résolution du problème suivant de G.Choquet : soient $\Omega$ un domaine de Green de $R^n$ , $z_o \in \partial\Omega$, $u \in R^n$ , $u \neq o$ et $C=\{z \in R^n ; \langle z-z_o, u\rangle > a.\|z-z_o\|, \|z-z_o\| < \rho\}$ un tronc de cône de révolution de sommet $z_o$ contenu dans $\Omega$ ; on peut se demander si toute suite de points sur l'axe du cône tendant vers $z_o$ est convergeante dans le compactifié de Martin (ordinaire) de $\Omega$ et dans ce cas si le point limite est minimal. La première question admet en général une réponse négative: plus précisemment, si on se donne un cône $C$ de demi-angle au sommet strictement inférieur à $\frac{\pi}{2}$ , on peut trouver un domaine $\Omega$ , $C$ et $z_o \in \partial\Omega$ mettant en défaut la propriété précédente (pour un contre-exemple) voir [ 1] , et la remarque 5 de [2]); lorsque $a$ est négatif ou nul, la situation est complètement différente et on peut établir par une variante de la méthode l'énoncé suivant : si $\Omega$ contient une boule $B(x,r)$ et si