Lecture Notes in Mathematics

1857

M. Émery M. Ledoux
M. Yor (Eds.)

Séminaire de Probabilités XXXVIII

Springer

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Editors

Michel Émery
Institut de Recherche
Mathématique Avancée
Université Louis Pasteur
7, rue René Descartes
67084 Strasbourg Cedex, France
e-mail: emery@math.u-strasbg.fr

Marc Yor
Laboratoire de Probabilités
et Modèles Aléatoires
Université Pierre et Marie Curie
Boîte courrier 188
4 place Jussieu
75252 Paris Cedex 05, France

Michel Ledoux
Laboratoire de Statistiques
et Probabilités
Université Paul Sabatier
118, route de Narbonne
31601 Toulouse Cedex, France
e-mail: ledoux@cict.fr

Library of Congress Control Number: 2004115715
Mathematics Subject Classification (2000): 60Gxx, 60Hxx, 60Jxx

ISSN 0075-8434 Lecture Notes in Mathematics
ISSN 0720-8766 Séminaire de Probabilités
ISBN 3-540-23973-1 Springer Berlin Heidelberg New York
DOI: 10.1007/b104072

Springer is a part of Springer Science + Business Media
http://www.springeronline.com

Printed in Germany

Typesetting: Camera-ready TEX output by the authors

41/3142/ du - 543210 - Printed on acid-free paper

C'est avec gratitude et admiration que nous dédions ce volume à

Jacques Azéma,

à l'occasion de son 65^e anniversaire. Ses travaux, parmi lesquels ceux sur le retournement du temps, le balayage, les fermés aléatoires et bien sûr la martingale d'Azéma, ont prolongé, toujours avec originalité et élégance, la théorie générale des processus.

Son apparente décontraction, sa réelle rigueur et ses incessantes questions ("his healthy skepticism", comme l'écrivait J. Walsh dans *Temps Locaux*), ont été indissociables du Séminaire de Probabilités pendant de nombreuses années.

We are also indebted and grateful to Anthony Phan, whose patient and time-consuming work behind the scene, up to minute details, on typography, formatting and TEXnicalities, was a key ingredient in the production of the present volume.

Volume XXXIX, which consists of contributions dedicated to the memory of P. A. Meyer, is being prepared at the same time as this one and should appear soon, also in the Springer LNM series. It may be considered as a companion to the special issue, also in memory of Meyer, of the *Annales de l'Institut Henri Poincaré.*

Finally, the Rédaction of the Séminaire is thoroughly modified: J. Azéma retired from our team after Séminaire XXXVII was completed; now, following his steps, two of us—M. Ledoux and M. Yor—are also leaving the board.

From volume XL onwards, the new Rédaction will consist of Catherine Donati-Martin (Paris), Michel Émery (Strasbourg), Alain Rouault (Versailles) and Christophe Stricker (Besançon). The combined expertise of the new members of the board will be an important asset to blend the themes which are traditionally studied in the Séminaire together with the newer developments in Probability Theory in general and Stochastic Processes in particular.

M. Émery, M. Ledoux, M. Yor

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Tanaka's Construction for Random Walks and Lévy Processes

Ronald A. Doney

Department of Mathematics, University of Manchester
Oxford Road, Manchester, UK M13 9PL
e-mail: rad@maths.man.ac.uk

Summary. Tanaka's construction gives a pathwise construction of "random walk conditioned to stay positive", and has recently been used in [3] and [8] to establish other results about this process. In this note we give a simpler proof of Tanaka's construction using a method which also extends to the case of Lévy processes.

1 The random walk case

If S is any rw starting at zero which does not drift to $-\infty$, we write S^* for S killed at time $\sigma := \min(n \geqslant 1 : S_n \leqslant 0)$, and $S^\uparrow$ for the for the harmonic transform of S^* which corresponds to "conditioning S to stay positive". Thus for $x > 0, y > 0$, and $x = 0$ when $n = 0$

$$P(S_{n+1}^\uparrow \in dy \,|\, S_n^\uparrow = x) = \frac{V(y)}{V(x)} P(S_{n+1} \in dy \,|\, S_n = x) = \frac{V(y)}{V(x)} P(S_1 \in dy - x), \tag{1}$$

where V is the renewal function in the weak increasing ladder process of $-S$. In [10], Tanaka showed that a process R got by time-reversing one by one the excursions below the maximum of S has the same distribution as $S^\uparrow$; specifically if $\{(T_k, H_k), k \geqslant 0\}$ denotes the strict increasing ladder process of S (with $T_0 = H_0 \equiv 0$) then R is defined by

$$R_0 = 0, \;\; R_n = H_k + \sum_{i=T_{k+1}+T_k+1-n}^{T_{k+1}} Y_i, \qquad T_k < n \leqslant T_{k+1}, \; k \geqslant 0. \tag{2}$$

If S drifts to $+\infty$, then it is well known (see [9]) that the post-minimum process

$$\overrightarrow{S} := (S_{J+n} - S_J, \; n \geqslant 0), \quad \text{where} \quad J = \max\Big\{n : S_n = \min_{r \leqslant n} S_r\Big\} \tag{3}$$

also has the distribution of $S^\uparrow$. In this case a very simple argument was given in [7] to show that the distributions of R and $\overrightarrow{S}$ agree, thus yielding a proof of

Tanaka's result in this case. The first point of this note is to show that a slight modification of this argument also yields Tanaka's result in the oscillatory case, without the somewhat tedious calculations in [10].

To see this, let S be any random walk with $S_0 \equiv 0$, $S_n = \sum_1^n Y_r$ for $n \geqslant 1$, introduce an independent Geometrically distributed random time G with parameter ρ and put $J_\rho = \max\{n \leqslant G : S_n = \min_{r \leqslant n} S_r\}$. In [7] a time-reversal argument was used to show that

$$(S_{J_\rho+n} - S_{J_\rho},\, 0 \leqslant n \leqslant G - J_\rho) \overset{D}{=} \left[\hat{\delta}_K(\rho), \dots, \hat{\delta}_1(\rho)\right] \qquad (4)$$
$$\overset{D}{=} \left[\hat{\delta}_1(\rho), \dots, \hat{\delta}_K(\rho)\right],$$

where $\hat{\delta}_1(\rho), \dots, \hat{\delta}_K(\rho)$ are the time reversals of the completed excursions below below the maximum of $\hat{S}(\rho) := (S_G - S_{G-n},\, 0 \leqslant n \leqslant G)$, and $[\dots]$ denotes concatenation. Note that the post-minimum process on the left in (4) has the same distribution as $(S_n,\, 0 \leqslant n \leqslant G \,|\, \sigma > G)$. Now in Theorem 1 of [4] it was shown that $S^\uparrow$ is the limit, in the sense of convergence of finite-dimensional distributions, of $(S_n,\, 0 \leqslant n \leqslant k \,|\, \sigma > k)$ as $k \to \infty$. (Actually [4] treated the case of conditioning to stay **non-negative**, and minor changes are required for our case). However it is easy to amend the argument there to see that as $\rho \downarrow 0$ this post-minimum process also converges in the same sense to $S^\uparrow$. Specifically a minor modification of Lemma 2 therein shows that

$$\liminf_{\rho \downarrow 0} \frac{P\{S_n \geqslant -x,\, n \leqslant G\}}{P\{S_n \geqslant 0,\, n \leqslant G\}} \geqslant V(x), \qquad x \geqslant 0,$$

and the rest of the proof is the same. Noting that $\hat{\delta}_1(\rho), \dots, \hat{\delta}_K(\rho)$ are independent and identically distributed and independent of K, and that $\hat{\delta}_1(\rho) \overset{D}{\to} \hat{\delta}_1$ and $K \overset{P}{\to} \infty$ as $\rho \downarrow 0$, we conclude that $S^\uparrow \overset{D}{=} [\hat{\delta}_1, \hat{\delta}_2, \dots] \overset{D}{=} R$, which is the required result.

2 The Lévy process case

The main point of this note is that, although the situation is technically more complicated, exactly similar arguments can be used to get a version of Tanaka's construction for Lévy processes.

We will use the canonical notation, and throughout this section $\mathbb{P}$ will be a measure under which the coordinate process $X = (X_t,\, t \geqslant 0)$ is a Lévy process which does not drift to $-\infty$ and is regular for $(-\infty, 0]$. For $x > 0$ we can use a definition similar to (1), with V replaced by the potential function for the decreasing ladder height subordinator to define a measure $\mathbb{P}_x^\uparrow$ corresponding to conditioning X starting from x to stay positive. But for $x = 0$ we need to employ a limiting argument. The following result is an immediate consequence of results in Bertoin [2]; see also Chaumont [5].

Theorem 1 (Bertoin). *Let τ be an* $\mathrm{Exp}(\rho)$ *random variable independent of* X*, and put*

$$J_\rho = \sup\{s < \tau : X_s = \underline{X}_s\} \quad \textit{where} \quad \underline{X}_s = \inf\{X_u : u < s\}.$$

Write $\underset{\rightarrow}{\mathbb{P}}^{(\rho)}$ *for the law of the post-minimum process* $\{X_{J_\rho} + s - X_{J_\rho}, 0 \leqslant s < \tau - J_\rho\}$ *under* $\mathbb{P}_0$*; then for each fixed* t *and* $A \in \mathcal{F}_t$

$$\lim_{\rho \downarrow 0} \underset{\rightarrow}{\mathbb{P}}^{(\rho)}\{A\} = P^{\uparrow}\{A\},$$

where $P^{\uparrow}$ *is a Markovian probability measure under which* X *starts at* 0 *and is such that the conditional law of* $X_{t+\cdot}$*, given* $X_t = x > 0$*, agrees with* $P_x^{\uparrow}$*.*

Remark 1. It has recently been shown that, under very weak assumptions, $P_x^{\uparrow}$ converges to $P^{\uparrow}$ as $x \downarrow 0$ in the sense of convergence of finite-dimensional distributions. See [6].

Next, we recall another result due to Bertoin which is the continuous time analogue of the result from [7] which we have stated as (4). Noting that (2) can be written in the alternative form

$$R_n = \bar{S}_{T_{k+1}} + \big(\bar{S} - S\big)_{T_k + T_{k+1} - n}, \qquad T_k < n \leqslant T_{k+1},$$

we introduce $\bar{X}_s = \sup_{u<s} X_u$ and

$$g(t) = \sup\big(s < t : X_s = \bar{X}_s\big), \qquad d(t) = \inf\big(s > t : X_s = \bar{X}_s\big),$$

the left and right endpoints of the excursion of $\bar{X} - X$ away from 0 which contains t, and define $R_t = \bar{X}_{d(t)} + \tilde{R}_t$, where

$$\tilde{R}_t = \begin{cases} \big(\bar{X} - X\big)_{(d(t)+g(t)-t)-} & \text{if } d(t) > g(t), \\ 0 & \text{if } d(t) = g(t). \end{cases}$$

We also introduce the future infimum process for X killed at time τ by

$$\underline{\underline{X}}_t = \inf\{X_s : t \leqslant s \leqslant \tau\},$$

and note that $\underline{\underline{X}}_0 = X_{J_\rho}$. The following result is established in the proof of Lemme 4 in [1]; note that, despite the title of the paper, this Lemme 4 is valid for any Lévy process which drifts to $+\infty$, and the result for the killed process, which is what we need is clearly valid for any Lévy process.

Theorem 2 (Bertoin). *Under* $\mathbb{P}_0$ *the law of* $\{(\tilde{R}_t, \bar{X}_{d(t)}), 0 \leqslant t < g(\tau)\}$ *coincides with that of*

$$\Big\{\Big(\big(X - \underline{\underline{X}}\big)_{J_\rho + t}, \underline{\underline{X}}_{J_\rho + t} - \underline{\underline{X}}_0\Big), 0 \leqslant t < \tau - J_\rho\Big\}.$$

Of course, an immediate consequence of this is the equality in law of

$$\{R_t, 0 \leqslant t < g(\tau)\} \quad \text{and} \quad \big\{X_{J_\rho + t} - \underline{\underline{X}}_0, 0 \leqslant t < \tau - J_\rho\big\}.$$

Letting $\rho \downarrow 0$ and appealing to Theorem 1 above we deduce

Theorem 3. *Under* $\mathbb{P}_0$ *the law of* $\{R_t, t \geqslant 0\}$ *is* $\mathbb{P}^{\uparrow}$*.*

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Some Excursion Calculations for Spectrally One-sided Lévy Processes

Ronald A. Doney

Department of Mathematics, University of Manchester
Oxford Road, Manchester, UK M13 9PL
e-mail: rad@maths.man.ac.uk

1 Introduction.

Let $X = (X_t, t \geqslant 0)$ be a spectrally negative Lévy process and write Y and $\widehat{Y}$ for the reflected processes defined by

$$Y_t = X_t - I_t, \qquad \widehat{Y}_t = S_t - X_t, \qquad t \geqslant 0,$$

where

$$S_t = \sup_{0 \leqslant s \leqslant t} (0 \vee X_s), \qquad I_t = \inf_{0 \leqslant s \leqslant t} (0 \wedge X_s).$$

In recent works by Avram, Kyprianou and Pistorius [1] and Pistorius [8] some new results about the times at which Y and $\widehat{Y}$ exit from finite intervals have been established. The proofs of these results in the cited papers involve a combination of excursion theory, Itô calculus, and martingale techniques, and the point of this note is to show that these results can be established by direct excursion theory calculations. These calculations are based on the known results for the two-sided exit problem for X in Bertoin [3], together with representations for the characteristic measures n and $\hat{n}$ of the excursions of Y and $\widehat{Y}$ away from zero. The representation for n has been established by Bertoin in [2] and that for $\hat{n}$ follows from results in Chaumont [4], (for a similar result for general Lévy processes see [5]), and are described in the next section.

2 Preliminaries

Throughout we assume that $X = (X_t, t \geqslant 0)$ is a Lévy process without positive jumps which is neither a pure drift nor the negative of a subordinator, and we adopt without further comment the notation of Chapter VII of [2]. In particular ψ and Φ denote the Laplace exponent of X and its inverse, and W denotes the scale function, the unique absolutely continuous increasing function with Laplace transform

$$\int_0^\infty \mathrm{e}^{-\lambda x} W(x)\,\mathrm{d}x = \frac{1}{\psi(\lambda)}, \qquad \lambda > \Phi(0).$$

The scale function determines the probability of X exiting at the top or bottom of a 2-sided interval, and the q-scale function $W^{(q)}$, which informally is the scale function of the process got by killing X at an independent Exp(q) time, determines also the distribution of the exit time. Specifically $W^{(q)}$ denotes the unique absolutely continuous increasing function with Laplace transform

$$\int_0^\infty \mathrm{e}^{-\lambda x} W^{(q)}(x)\,\mathrm{d}x = \frac{1}{\psi(\lambda) - q}, \qquad \lambda > \Phi(q),\ q \geqslant 0, \tag{1}$$

and for convenience we set $W^{(q)}(x) = 0$ for $x \in (-\infty, 0)$. We also need the "adjoint scale function" defined by $Z^{(q)}(x) = 1$ for $x \leqslant 0$ and

$$Z^{(q)}(x) = 1 + q\int_0^x W^{(q)}(y)\,\mathrm{d}y \qquad \text{for } x > 0. \tag{2}$$

Extending previous results due to Emery [6], Takacs [11], Rogers [9], and Suprun [10], in [3] Bertoin gave the full solution to the 2-sided exit problem in the following form:

Proposition 1. *Define for $a \geqslant 0$ the passage times*

$$T_a = \inf(t \geqslant 0 : X_t > a), \qquad \widehat{T}_a = \inf(t \geqslant 0 : -X_t > a).$$

Then for $0 \leqslant x \leqslant a$ we have

$$\mathbb{E}_x\left(\mathrm{e}^{-qT_a}; T_a < \widehat{T}_0\right) = \frac{W^{(q)}(x)}{W^{(q)}(a)}, \tag{3}$$

and

$$\mathbb{E}_x\left(\mathrm{e}^{-q\widehat{T}_0}; \widehat{T}_0 < T_a\right) = Z^{(q)}(x) - \frac{W^{(q)}(x)Z^{(q)}(a)}{W^{(q)}(a)}. \tag{4}$$

Furthermore let $U^{(q)}$ denote the resolvent measure of X killed at the exit time $\sigma_a := T_a \wedge \widehat{T}_0$; then $U^{(q)}$ has a density which is given by

$$u^{(q)}(x,y) = \frac{W^{(q)}(x)}{W^{(q)}(a)} W^{(q)}(a-y) - W^{(q)}(x-y), \qquad x,\, y \in [0,a). \tag{5}$$

Remark 1. Suppose that $\mathbb{P}^a$ is a measure under which X is a Lévy process having the same characteristics as under $\mathbb{P}$ except that Π is replaced by

$$\Pi^a(\mathrm{d}x) = \Pi(\mathrm{d}x)\,\mathbf{1}_{\{x \geqslant -a\}} + \Pi\big((-\infty, -a)\big)\,\delta_{-a}(\mathrm{d}x),$$

where $\delta_{-a}(\mathrm{d}x)$ denotes a unit mass at $-a$. Then it is clear that up to time σ_a, X behaves the same under $\mathbb{P}^a$ as it does under $\mathbb{P}$. Thus the righthand sides of (3) and (4) are unchanged if $W^{(q)}$ is replaced by $W^{(q)a}$, the scale function for X under $\mathbb{P}^a$. It then follows that we must have the identity

$$W^{(q)}(x) \equiv W^{(q)a}(x) \qquad \text{for } 0 \leqslant x \leqslant a.$$

Note that the behaviour of Y and $\widehat{Y}$ up to the time that they exit the interval $[0, a]$ is also the same under $\mathbb{P}^a$ as it is under $\mathbb{P}$.

Remark 2. The probability measures $\mathbb{P}$ and $\mathbb{P}^\#$ are said to be *associates* if X is also a spectrally negative Lévy process under $\mathbb{P}^\#$ and there is a constant $\delta \neq 0$ such that

$$\mathbb{P}^\#(X_t \in \mathrm{d}x) = \mathrm{e}^{\delta x}\, \mathbb{P}(X_t \in \mathrm{d}x), \qquad -\infty < x < \infty.$$

It is known that if X drifts to $-\infty$ under $\mathbb{P}$ then $\mathbb{P}^\#$ exists, is unique, and $\delta = \Phi(0) > 0$ is a zero of ψ. (See [2], p. 193.) On the other hand, if X drifts to ∞ under $\mathbb{P}$ then $\mathbb{P}^\#$ may or may not exist; if it does it is unique, and δ is a negative zero of ψ. In both cases the corresponding scale functions are related by $W^\#(x) = \mathrm{e}^{\delta x} W(x)$. Note that if the Lévy measure is confined to a finite interval, as Π^a is in Remark 1, then $\psi(\lambda)$ exists for all real λ, and if $\mathbb{E}\, X_1 \neq 0$ then it has 2 real zeros, so the associate measure exists.

We also need some information about the excursion measures n and $\hat{n}$ of Y and $\widehat{Y}$ away from zero. (n.b. this notation is the opposite of that in [2]). In what follows it should be noted that whereas 0 is always regular for $(0, \infty)$ for Y, it is possible for 0 to be irregular for $(0, \infty)$ for $\widehat{Y}$. (This situation was excluded in [8].) In this case we adopt the convention outlined on p. 122 of [2], which allows us to assume that $\widehat{Y}$ has a continuous local time at 0.

In the following result ζ denotes the lifetime of an excursion and $\mathbb{Q}_x$ and $\mathbb{Q}^*_x$ denote the laws of X and $-X$ killed on entering $(-\infty, 0)$ respectively.

Proposition 2. *Let $A \in \mathcal{F}_t, t > 0$, be such that $n(A^o) = 0$ (respectively $\hat{n}(A^o) = 0$), where A^o is the boundary of A with respect to the J-topology on D. Then there are constants k and $\hat{k}$ (which depend only on the normalizations of the local time at zero of Y and $\widehat{Y}$) such that*

$$n(A,\, t < \varsigma) = k \lim_{x \downarrow 0} \frac{\mathbb{Q}_x(A)}{W(x)}\,, \tag{6}$$

and, assuming further that if X drifts to $+\infty$ under $\mathbb{P}$ then the associate measure $\mathbb{P}^\#$ exists,

$$\hat{n}(A,\, t < \varsigma) = \hat{k} \lim_{x \downarrow 0} \frac{\mathbb{Q}^*_x(A)}{x}\,. \tag{7}$$

Proof. According to Propositions 14 and 15, p. 201–202 of [2] for any $A \in \mathcal{F}_t$ we have

$$n(A,\, t < \varsigma) = k\,\mathbb{E}^{\uparrow}\left(W(X_t)^{-1};\, A\right), \tag{8}$$

where $\mathbb{P}^{\uparrow}$ is the weak limit in the Skorohod topology as $x \downarrow 0$ of the measures $\mathbb{P}^{\uparrow}_x$ which correspond to "conditioning X to stay positive", and are defined by

$$\mathbb{P}^{\uparrow}_x(X_t \in \mathrm{d}y) = \frac{W(y)}{W(x)}\,\mathbb{Q}_x(X_t \in \mathrm{d}y), \qquad x > 0,\, y > 0.$$

Combining these results and using the assumption on A gives (6). The proof of (7) is similar. If X does not drift to $+\infty$ under $\mathbb{P}$ the potential function of the increasing ladder height process is given by

$$V(x) = \begin{cases} x & \text{if } \mathbb{E}\,X_1 = 0, \\ 1 - \mathrm{e}^{-x\Phi(0)} & \text{if } \mathbb{E}\,X_1 < 0, \end{cases}$$

so that $V(x) \backsim cx$ as $x \downarrow 0$ in both cases. The analogue of (8) is

$$\hat{n}(A,\, t < \varsigma) = k\,\mathbb{E}^{*\uparrow}\left(V(X_t)^{-1};\, A\right)$$

where, by Theorem 6 of [4], $\mathbb{P}^{*\uparrow}$ is the weak limit of the measures

$$\mathbb{P}^{*\uparrow}_x(X_t \in \mathrm{d}y) = \frac{V(y)}{V(x)}\,\mathbb{Q}^{*}_x(X_t \in \mathrm{d}y), \qquad x > 0,\, y > 0.$$

If X does drift to $+\infty$ under $\mathbb{P}$ then it is easy to check that, with $\varepsilon = (\varepsilon(t),\, t \geqslant 0)$ denoting a generic excursion and $\hat{n}^{\#}$ denoting the excursion measure of $\widehat{Y}$ under the associate measure $\mathbb{P}^{\#}$,

$$\hat{n}\left(A, \varepsilon(t) \in \mathrm{d}y,\, t < \varsigma\right) = \mathrm{e}^{-\delta y}\,\hat{n}^{\#}\left(A,\, \varepsilon(t) \in \mathrm{d}y,\, t < \varsigma\right). \tag{9}$$

Since X drifts to $-\infty$ under $\mathbb{P}^{\#}$ we can apply the previous result and the fact that

$$\mathbb{Q}^{*}_x(X_t \in \mathrm{d}y) = \mathrm{e}^{-\delta(y-x)}\,\mathbb{Q}^{\#*}_x(X_t \in \mathrm{d}y)$$

to complete the proof. □

Remark 3. One way to check (9) is to use our knowledge of the Wiener–Hopf factors and equation (7), p. 120 of [2] to compute the double Laplace transforms of $\hat{n}(\varepsilon(t) \in \mathrm{d}y,\, t < \varsigma)$ and $\mathrm{e}^{-\delta y}\hat{n}^{\#}(\varepsilon(t) \in \mathrm{d}y,\, t < \varsigma)$.

We also need some facts about $W^{(q)}$:

Lemma 1. (i) $\lim_{x\downarrow 0} \frac{W^{(q)}(x)}{W(x)} = 1$;

(ii) *If X has unbounded variation then $W^{(q)\prime}(x)$, the derivative with respect to x of $W^{(q)}(x)$ exists and is continuous for all $x > 0$.*

(iii) *If X has bounded variation let $\mathcal{D}$ denote $\{x : \Pi$ has positive mass at $-x\}$. Then $W_+^{(q)'}(x)$ and $W_-^{(q)'}(x)$, the right and lefthand derivatives of $W^{(q)}(x)$ exist at all $x > 0$, agree off $\mathcal{D}$, and*

$$\lim_{y \downarrow x} W_+^{(q)'}(y) = W_+^{(q)'}(x) \qquad \textit{for all } x \in \mathcal{D}.$$

Proof. (i) This follows from the expansion

$$W^{(q)}(x) = \sum_{k=1}^{\infty} q^{k-1} W^{(k*)}(x), \tag{10}$$

where $W^{(k*)}$ denotes the k-fold convolution of W, together with the bound

$$W^{(k*)}(x) \leqslant \frac{x^{k-1} W(x)^k}{(k-1)!}, \qquad k \geqslant 1,\, x \geqslant 0.$$

(ii) Provided X does not drift to $-\infty$ under $\mathbb{P}$, we have the representation

$$W(x) = c \exp\Big(- \int_x^{\infty} \hat{n}(h(\varepsilon) > t)\, dt \Big),$$

(see [2], p. 195). As pointed out in [8], this implies that

$$W'_+(x) = W(x)\, \hat{n}\big(h(\varepsilon) > x\big), \qquad W'_-(x) = W(x)\, \hat{n}\big(h(\varepsilon) \geqslant x\big),$$

and the result follows when $q = 0$ since $\hat{n}$ has no atoms in the case of unbounded variation, (see [7]). If X does drift to $-\infty$ under $\mathbb{P}$ we use the device of the associate measure $\mathbb{P}^{\#}$ introduced in Remark 2. Since X drifts to ∞ and has unbounded variation under $\mathbb{P}^{\#}$, it is easy to check that the result also holds in this situation. The case when $q > 0$ again follows easily, using (10).

(iii) In this case excursions of $\widehat{Y}$ away from 0 start with a jump, and then evolve according to the law of $-X$. Since 0 is irregular for $(-\infty, 0)$, this shows that $\hat{n}(h(\varepsilon) = x) > 0$ for all $x \in \mathcal{D}$, but the fact that X has an absolutely continuous resolvent means that $\hat{n}(h(\varepsilon) = x) = 0$ for all $x \notin \mathcal{D}$, and this implies the stated results for $q = 0$. Again the results for $q > 0$ follow easily, using (10). □

To demonstrate the use of the above result, we calculate below the n and $\hat{n}$ measures of a relevant subset of excursion space. Put $h(\varepsilon) := \sup_{t<\varsigma} \varepsilon(t)$ and $T_a(\varepsilon) = \inf\{t : \varepsilon(t) > a\}$ for the height and the first passage time of a generic excursion ε whose lifetime is denoted by $\varsigma(\varepsilon)$, and with Λ_q denoting an independent Exp(q) random variable set $A = B \cup C$, where

$$B = \{\varepsilon : h(\varepsilon) > a,\, T_a(\varepsilon) \leqslant \varsigma(\varepsilon) \wedge \Lambda_q\} \quad \text{and} \quad C = \{\varepsilon : h(\varepsilon) \leqslant a,\, \Lambda_q < \varsigma(\varepsilon)\}.$$

Since we will only be concerned with ratios of n and $\hat{n}$ measures in the following we will assume that $k = \hat{k} = 1$.

Lemma 2. *In all cases*

$$\alpha := n(A) = \frac{Z^{(q)}(a)}{W^{(q)}(a)}, \tag{11}$$

and, provided that if X drifts to $+\infty$ under $\mathbb{P}$ then the associate measure $\mathbb{P}^{\#}$ exists,

$$\hat{\alpha} := \hat{n}(A) = \frac{W_+^{(q)'}(a)}{W^{(q)}(a)}. \tag{12}$$

Proof. Since ([2], p. 202)

$$n\big(h(\varepsilon) > x\big) = c/W(x)$$

is continuous, we see from (6) that

$$n\big(h(\varepsilon) > a,\, T_a(\varepsilon) \in \mathrm{d}t\big) = \lim_{x\downarrow 0} \frac{\mathbb{Q}_x\{T_a \in \mathrm{d}t\}}{W(x)} = \lim_{x\downarrow 0} \frac{\mathbb{P}_x\{T_a < \widehat{T}_0,\, T_a \in \mathrm{d}t\}}{W(x)},$$

and

$$n(\varepsilon_t \in \mathrm{d}y,\, T_a(\varepsilon) > t) = \lim_{x\downarrow 0} \frac{\mathbb{Q}_x\{X_t \in \mathrm{d}y,\, T_a > t\}}{W(x)} = \lim_{x\downarrow 0} \frac{\mathbb{P}_x\{X_t \in \mathrm{d}y,\, \sigma_a > t\}}{W(x)}.$$

Thus

$$\begin{aligned}\alpha = n(B) + n(C) &= \lim_{x\downarrow 0} \frac{1}{W(x)} \Big(\mathbb{E}_x\{\mathrm{e}^{-qT_a};\, T_a < \widehat{T}_0\} + \mathbb{P}_x\{\Lambda_q < \sigma_a\}\Big) \\ &= \lim_{x\downarrow 0} \frac{1}{W(x)} \Big(1 - \mathbb{E}_x\{\mathrm{e}^{-q\widehat{T}_0}; \widehat{T}_0 < T_a\}\Big).\end{aligned}$$

Combining this with (4) gives

$$\alpha = \lim_{x\downarrow 0} \frac{1 - Z^{(q)}(x)}{W(x)} + \frac{Z^{(q)}(a)}{W^{(q)}(a)} \lim_{x\downarrow 0} \frac{W^{(q)}(x)}{W(x)} = \frac{Z^{(q)}(a)}{W^{(q)}(a)}.$$

Note next that the results of Lemma 2.3 show that the right-hand side of (12) is a cadlag function of a; since it is easy to see that the same is true of the left-hand side, it suffices to establish these results for $a \notin \mathcal{D}$. In this case we have $\hat{n}(h = a) = 0$, so the required J-continuity holds and by a similar argument we can use (7) and (3) to get

$$\begin{aligned}\hat{a} &= \lim_{x\downarrow 0} \frac{1}{x} \Big(\mathbb{E}_{a-x}\{\mathrm{e}^{-q\widehat{T}_0};\, \widehat{T}_0 < T_a\} + \mathbb{P}_{a-x}\{\Lambda_q < \sigma_a\}\Big) \\ &= \lim_{x\downarrow 0} \frac{1}{x} \Big(1 - \mathbb{E}_{a-x}\{\mathrm{e}^{-qT_a};\, T_a < \widehat{T}_0\}\Big) \\ &= \lim_{x\downarrow 0} \frac{1}{x} \left(\frac{W^{(q)}(a) - W^{(q)}(a-x)}{W^{(q)}(a)}\right) = \frac{W_+^{(q)'}(a)}{W^{(q)}(a)}.\end{aligned}$$

□