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MICHAEL J. PANIK

FUNDAMENTALS OF CONVEX ANALYSIS

Duality, Separation, Representation, and Resolution

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by

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Preface

0.1. An Overview

My objective in writing this book is to offer to students of economics, management science, engineering, and mathematics an in-depth look at some of the fundamental features of a particular subset of general nonlinear mathematical analysis called ***convex analysis***. While the spectrum of topics constituting convex analysis is extremely wide, the principle themes which will be explored herein are those of ***duality, separation, representation, and resolution***. In this regard, these broad topic areas might even be referred to as the mathematical foundations or basic building blocks of convex analysis. Indeed, one could not reasonable expect to address, with even a modicum of success, the theoretical aspects of matrix games, optimization, and general equilibrium analysis without them.

A theme which occupies a key position in the area of convex analysis is that of ***duality***. This property asserts that convex structures have a dual description, *i.e.*, corresponding to each convex object A in a space of finite dimension there is a dual object B belonging to the same space. In this regard, if B is given, then A can be uniquely generated. For instance, given a finite cone $\mathcal{C}$, the cone polar to $\mathcal{C}$ is $\mathcal{C}^+ = \{\mathbf{y} \mid \mathbf{y}'\mathbf{x} \geq 0, \mathbf{x} \in \mathcal{C}\}$. By taking the polar of $\mathcal{C}^+$ we recover $\mathcal{C}$ itself so that we may write $\mathcal{C} \leftrightarrow \mathcal{C}^+$ or $\mathcal{C} = (\mathcal{C}^+)^+$. This specification of duality has its roots in the notion of ***separation***, *e.g.*, under a given set of assumptions it is possible to separate (disjoint) convex sets by a hyperplane. In fact, most of the duality theorems encountered in this work are proved by employing virtually the same set of separation arguments.

Also playing an important role in convex analysis is the idea of ***representation***, *i.e.*, there can exist proper subsets of elements of a convex set which possess all the information about the original convex set itself. For example, knowing only the collection of extreme points of a convex polytope allows us to fully determine the entire set by forming the convex hull of the set of extreme points. In a similar vein, the theme of ***resolution***, drawing upon a finite basis argument, serves to decompose the elements within a given convex set into the convex hulls (or conical hull as appropriate) of a pair of disjoint finite point sets. That is, a point in a particular convex set can, at times, be represented as the (vector) sum of points from two different convex structures. For instance, a convex polyhedron is resolvable into the sum of a polytope and a polyhedral cone.

Chapter 1 lays out the mathematical prerequisites. It is assumed that the reader has had some minimal exposure to set theory as well as linear algebra and matrices. This chapter begins with a review of essential concepts such as: the Euclidian norm; linear dependence and independence; spanning set, basis, and dimension; solutions sets for simultaneous linear systems (treating both the homogeneous and nonhomogeneous cases along with the detection of inconsistencies); and linear subspaces and their duals. It then moves into the realm of point-set-theory and defines in considerable detail notions such as; neighborhoods; open and closed sets; points of closure and accumulations; convergence (in norm); compactness criteria; a linear variety or affine set; affine hull; and affine independence.

In chapter 2 we consider the structure of convex point sets. Specifically, the definition of a convex combination and convex set proper are given along with the essential properties of the latter. Also included are: Helly's theorem (1921); Berge's theorem (1959, 1963); the concept of a convex hull; Carathéodory's theorem (1970); and the relative interior of a convex set. Convex sets and their properties are at the heart of convex analysis and will be used extensively in the remainder of the text.

Chapter 3 begins with a detailed discussion of hyperplanes and their associated open and closed half-planes. After defining the characteristics of weak, proper, strict, and strong separation, a set of theorems is advanced which treats a variety of types of separation between a point and a set and between two convex sets. In particular, these theorems posit conditions for the "existence" of separating hyperplanes. Of special interest is the particular methodology used to execute most proofs of separation theorems. As will be readily demonstrated in this chapter, the usual ***modus operandi*** for dealing with the separation of two convex sets $\mathscr{S}_1, \mathscr{S}_2$ consists of redefining the problem as one involving the separation of the origin from the convex set $\mathscr{S}_1 - \mathscr{S}_2$.

An additional set of theorems pertaining to the existence of supporting or tangent hyperplanes is also included. Such theorems are important because they allow us to characterize or "represent" a convex set $\mathscr{S}$ in terms of its associated (finite) set of supporting hyperplanes at boundary points on $\mathscr{S}$. A representation of this type can be thought of as being of the ***external*** or outer variety.

The concept of separation holds a preeminent position in convex analysis in that it serves as a major input in deriving certain ***theorems of the alternative.*** A few of these theorems (involving disjoint alternatives framed in terms of linear equalities and/or inequalities) are introduced in chapter 3 as an application of the separation concept. In particular, the theorems of Farkas (1902) and Gordan (1873) along with one of Gale's theorems (1960) (dealing with the existence of a nonnegative solution to a system of linear inequalities) are all developed and illustrated with the use of a strong separation theorem. As the reader will soon note, Farkas' theorem will be used extensively throughout this work. In fact, it is actually introduced quite early in chapter 1 as the basis for the specification of a criterion used in detecting whether or not a simultaneous linear equation system is inconsistent.

Chapter 4 deals with the concepts of convex cones and finite cones, with the latter also classified as convex. These structures serve the essential function of geometrically illustrating the solution sets for homogeneous systems of linear equalities/inequalities. Also introduced are negative, orthogonal, dual, polar, normal, support, and barrier cones along with the process of determining the dimension of a finite cone. The concept of a ray or half-line is used extensively as an element in the construction of finite cones; its dual and polar serve to specify closed half-spaces. Properties of convex and finite cones are fully explored and the duality property of finite cones is amplified to theorem status, *i.e.*, a duality theorem for finite cones is proven using Farkas' theorem (in fact, it is demonstrated that the duality theorem is equivalent to Farkas' theorem) and then directly related to the concept of strong separation.

In order to explore the various (equivalent) ways of representing or generating a finite cone, the concepts of a conical combination and conical hull are introduced along with Carathéodory's theorem for cones. Also introduced are the notions of: extreme vectors (as well as extreme half-lines and half-spaces); semi-positively independent set of vectors; extreme supporting hyperplanes and half-planes; and extreme solutions of homogeneous linear inequalities. All of this definitional material lends support to the specification of a set of theorems dealing with the structure of finite cones, *e.g.*, we explore; Minkowski's theorem (1910) (the intersection of finitely many half-spaces is a conical combination of finitely many generators); a second theorem which states essentially that a finite cone can be generated by using only its set of extreme vectors; and Weyl's theorem (1935, 1950) (the set of conical combinations of a finite set of vectors corresponds to the intersection of a finite number of extreme supporting half-spaces). In fact, the theorems of Minkowski and Weyl serve to establish the so-called ***sum cone and intersection cone equivalence.*** It is also observed that Weyl's theorem implies the duality theorem for finite cones (*i.e.*, Farkas' theorem) we well as Minkowski's theorem. Moreover, the theorems of Minkowski and Farkas in combination render an "indirect" proof of Weyl's theorem.

Chapter 4 ends with a discussion of a couple of separation theorems for convex cones. These theorems are then used to obtain an representation theorem for a closed convex cone (*i.e.*, any such cone is the intersection of the set of homogeneous closed half-spaces which contain it). In addition, we establish the notion that Farkas' theorem can be cast in terms of finite cones and then interpreted as a separation theorem for a cone and an individual vector or a cone and an open half-space.

An important question often encountered in convex analysis is whether or not certain dual homogeneous linear systems possess a solution. This is the subject matter of chapter 5. Here we consider pairs of finite systems of homogeneous linear equalities and/or inequalities in which the variables are either nonnegative or unrestricted in sign. Moreover, these systems are structured in a fashion such that there is a one-to-one correspondence between unrestricted variables in one system and equations in the other and between nonnegative variables in one system and inequalities in the other. Under the aforementioned correspondence one can pass from a given system of homogeneous linear inequalities and/or equalities involving nonnegative and/or unrestricted variables to a second such system and conversely.

The chapter begins with a lemma by Tucker (1956) for dual homogeneous linear relations exhibiting a special positivity property and then moves into the analysis of a battery of Tucker's existence theorems (1956) for similar pairs of dual systems. An additional set of existence theorems is addressed which provides the foundation for the development of the concept of ***complementary slackness*** in pairs of dual systems and in a specialized self-dual system. Much of the material appearing in this chapter lends itself to applications in the area of linear programming (especially where questions of the existence and uniqueness of solutions as well as their feasibility are concerned).

The material developed in chapters 3-5 serves as the cornerstone for the treatment of ***theorems of the alternative*** for linear systems presented in chapter 6. Such theorems involve two mutually exclusive systems of linear inequalities and/or equalities denoted as, say, (I) and (II). They then assert that either system (I) has a solution or system (II) has a solution, but never both. In addition, a ***transposition theorem***, which is a special type of theorem of the alternative, considers the disjoint alternatives of solvability or contradiction given that in one system a vector is a linear combination of vectors from the other. In fact, a transposition theorem can be viewed as the algebraic counterpart of a separation theorem.

A whole host of important theorems of the alternative are discussed and, as is appropriate, interpreted geometrically. Specifically, the theorems included are those of: Slater (1951); Tucker (1956); Motzkin (1936); Gordan (1873); Steimke (1915); Farkas (1902); Gale (1960); von Neumann (1944); and Mangasarian (1969). Moreover, these theorems cover both homogeneous as well as nonhomogeneous systems and consider solutions which may be characterized as positive, nonnegative, semi-positive, or restricted to a convex combination. The material offered in this chapter lends itself to a wealth of applications in the areas of game theory and mathematical programming (*i.e.*, the specification of first-order optimality conditions in the presence of constraints; nondifferentiable optimization; constraint qualifications, etc.).

The principal focus of chapter 7 is the determination of what are called ***basic solutions*** (as well as ***basic feasible solutions***) to systems of nonhomogeneous linear equalities. After defining a basic solution, a step-wise procedure for obtaining a set of basic variables is outlined and accompanied by the process of "swapping" one basis vector for another so as to obtain a different basic feasible solution. Here too a step-by-step summary algorithm is reported along with several detailed examples which serve to illustrate the salient features of the calculations involved. Also included is a discussion on the circumstances under which we can find at least one basic (feasible) solution to a linear equation system.

Having developed the concept of a basic feasible solution to a linear equation system, this chapter next explores the structure of the solution set for the same. Let this set be denoted as $\mathscr{S} = \{\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geqq \mathbf{0},\ \mathbf{x} \in \mathbf{R}^n\}$. After specifying the form of the homogeneous system associated with $\mathscr{S}$, a ***resolution theorem*** is established which essentially states that any vector in $\mathscr{S}$ is expressible as a convex combination of the set of basic feasible solutions plus a homogeneous solution associated with the linear system. We next examine extreme homogeneous solutions and then generalize the preceding resolution theorem to take account of such solutions. Both resolution theorems are structured so that $\mathscr{S}$ may or may not be bounded.

The next topic in Chapter 7 deals with the development, via Farkas' theorem, of a set of conditions which indicate when a mixed simultaneous system of linear inequalities and equalities has no solution. After this comes a section on complementary slackness in pairs of dual systems. In this regard, sets of weak and strong complementary slackness conditions are derived for general dual homogeneous systems; along with these, similar conditions, in the form of a set of complementary inequalities, are derived for a specialized self-dual system.

Chapter 8 begins with a discussion of extreme points and extreme direction for general convex sets. It addresses the conditions which underlie the existence of extreme points as well as the representation of a convex set in terms of its collection of extreme points (*i.e.*, we examine a theorem which states that, under certain conditions, extreme points form a minimal subset whose convex hull equals the set itself). Next considered are the concepts of: recession and extreme directions of a convex set; recession and extreme half-lines; and the convex cone generated by a convex set. The extension of the convex hull and affine hull concepts to sets which consist of both points and recession directions is also included. This enables us to offer expanded versions of the preceding representation theorem (a convex set can be expressed as the convex hull of its set of extreme points and extreme directions) and Carathédory's theorem (also framed in terms of points and directions).

We next study what are called "faces" of ***polyhedral convex sets*** (*e.g.*, extreme points, facets, and edges are all types of faces). Before doing so, however, it is important to determine what it actually means for a convex set to be characterized as "polyhedral." Following Rockafellar (1970), the property of being polyhedral reflects the notion that a given set is the intersection of finitely many closed half-spaces, *i.e.*, it is the solution set to some finite system of linear inequalities. (Note that we obtain what is termed a ***polyhedral convex cone*** if the said system is homogeneous,*i.e.*, all bounding hyperplanes pass through the origin.) In this regard, the quality of being polyhedral imposes a "finiteness" condition on the outer or ***external representation*** of a convex set (an n-dimensional closed convex set is the intersection of its set of supporting or tangent closed half-spaces). Dually, a finiteness condition can also be placed on the ***internal representation*** of a convex set (here a polyhedral convex set can be represented as the convex hull of its set of extreme points plus the conical combination of its set of extreme directions). We may note further that if the polyhedral convex sets under discussion are bounded, then they are called ***convex polytopes.*** Relative to the discussion of polyhedral faces mentioned above, the topics covered are: degenerate and adjacent extreme points; the dimension, minimual representation, and affine hull of a convex polyhedron; and proper face structures.

The preceding bit of material on convex polyhedra, along with some of the developments in chapter 7, now sets the stage for determining the location of extreme points. In particular, we posit a necessary and sufficient characterization of an extreme point by demonstrating that there exists a one-to-one correspondence between basic feasible solutions to the equalities defining a convex polyhedron and the extreme points of the polyhedron. This is then followed by an existence theorem (for extreme points) and a representation theorem which states that every convex polytope is the convex hull of its set of extreme points and conversely.

The definition of a recession direction and a recession cone appears next. It is then shown that, under certain conditions, a recession direction for a convex polyhedron is also an extreme direction. After examining a set of unboundedness criteria for convex polyhedra, the discussion turns to the development of an existence theorem for extreme directions.

At this point we are now able to offer a combined extreme point and extreme direction representation theorem for polyhedral convex sets. This is followed by an analysis of the resolution or decomposition of convex polyhedra. In particular, after stating a finite basis theorem for polyhedra, it is demonstrated that every polyhedral convex set is resolvable into the sum of a bounded convex polyhedron (or polytope) and a polyhedral convex cone. This is then followed by a finite bases theorem for polytopes.

The next section appearing in chapter 8 involves the separation of convex polyhedra. Here we extend some of the fundamental separation results for convex sets developed in chapter 3 to the case where at least one of the sets being separated is a convex polyhedron.

In chapter 9 we develop the notion of a k-dimensional simplex along with the definition of a standard n-simplex and unit simplex. Simplicial faces, facets, and carrier faces are next introduced along with concepts such as a simplicial complex and a simplicial decomposition (triangulation). Additional definitions such as a subdivision, and in particular the barycentric subdivision of a simplex, a simplicial mapping, and an (integer) labeling function set the stage for the development of Sperner's lemma, the Knaster-Kuratowski-Mazurkiewicz (K-K-M) theorem, Brouwer's (fixed point) theorem (along with a modification by Schauder), and Kakutani's (fixed point) theorem. Specifically, Sperner's lemma informs us that a properly labeled simplex contains an odd number of completely labeled subsimplexes. the K-K-M theorem provides us with a set of conditions which guarantee that the intersection of a collection of closed sets on a simplex is nonempty. And Brouwer's theorem demonstrates that a continuous point-to-point mapping of a simplex into itself admits at least one fixed point, *i.e.*, a point which is transformed into itself under the mapping. It is further shown that these three theorems are mathematically equivalent. Finally, Kakatani's theorem, which is a generalization of Brouwer's theorem to multivalued functions, states that an upper hemicontinuous point-to-set mapping of a compact convex set into itself has a fixed point.

0.2. A Note on the Method of Mathematical Induction

Quite often in mathematical analysis there are theorems which can be formulated in terms of "n" in that they assert a certain equation or proposition holds where n is any positive integer. For theorems such as these, an appropriate method of proof is ***mathematical induction.*** This procedure consists of the following two steps:

(1) Verify that the theorem/proposition holds for $n = 1$ (usually);

(2) Assume that the theorem/proposition holds for $n = p$ (the ***induction hypothesis***) and then prove that it holds for $n = p+1$.

Clearly the process of mathematical induction involves a type of "domino effect," *i.e.* once the proposition is proved for a particular integer, then the proposition will automatically follow for the next integer, and the next one, and the next, and so on ***ad infinitum.*** Given that steps 1, 2 have been executed, the "chain reaction" inherent in the process is set in motion and subsequently applies for any $n > 0$.

Suppose steps 1, 2 have been carried out for some theorem which is to be proved. How can we be sure that this procedure actually proves the theorem? To answer this let us assume, to the contrary, that the theorem under consideration is false. In this instance there exist positive integers for which the theorem is false and thus there must be some smallest integer, say $M+1$, for which the theorem is false. Since Step 1 precludes the integer $M+1$ from equaling 1, there is an integer M preceding $M+1$ and, for $n = M$, the theorem is true. Then under step 2, the theorem follows for $n = M+1$ and thus a contradiction occurs. But then this means that the assumption "the theorem is false" was incorrect and thus the theorem must be true.

0.3. Vector Notation

In the material which follows we shall deal with n-dimensional vector spaces taken over a field $\mathbf{R}$ of real scalars. Since the elements in $\mathbf{R}$ are "ordered," concepts such as "positive, semipositive, or nonnegative" can be defined. Specifically, the relations " $>$, $\geq$, and $\geqq$ " constitute a partial ordering on the vectors in $\mathbf{R}^n$, *i.e.*, for

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_n \end{bmatrix} \in \mathbf{R}^n \quad \text{or} \quad \mathbf{x} = (x_i) \in \mathbf{R}^n,\ i = 1, \ldots, n,$$

(1) $\mathbf{x}$ is a ***strictly positive vector*** (written $\mathbf{x} > \mathbf{0}$) if $x_i > 0$ for all i;

(2) $\mathbf{x}$ is a ***nonnegative vector*** (denoted $\mathbf{x} \geqq \mathbf{0}$) if $x_i \geq 0$ for all i, and

(3) $\mathbf{x}$ is a ***semipositive vector*** (written $\mathbf{x} \geq \mathbf{0}$) if $\mathbf{x} \geqq \mathbf{0}$ but $\mathbf{x} \neq \mathbf{0}$, *i.e.*, $\mathbf{x}$ has at least one positive component.

Moreover, for vectors $\mathbf{x}_1, \mathbf{x}_2 \in \mathbf{R}^n$, we may write $\mathbf{x}_1 > \mathbf{x}_2$, $\mathbf{x}_1 \geqq \mathbf{x}_2$, or $\mathbf{x}_1 \geq \mathbf{x}_2$ according to whether $\mathbf{x}_1 - \mathbf{x}_2$ is strictly positive, nonnegative, or semipositive.

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