SHAYLE R. SEARLE

MATRIX ALGEBRA USEFUL FOR STATISTICS

WILEY SERIES IN PROBABILITY
AND MATHEMATICAL STATISTICS

Matrix Algebra Useful for Statistics

SHAYLE R. SEARLE

Professor of Biological Statistics
Biometrics Unit
New York State College of Agriculture and Life Sciences
Cornell University,
Ithaca, N.Y.

JOHN WILEY & SONS

New York • Chichester • Brisbane • Toronto • Singapore

Library of Congress Cataloging in Publication Data:

Searle, S. R. (Shayle R.), 1928–
Matrix algebra useful for statistics.

(Wiley series in probability and mathematical statistics. Applied probability and statistics, ISSN 0271-6356)
Bibliography: p.
Includes index.
1. Matrices. 2. Statistics. I. Title. II. Series.

QA188.S4 512.9'434 82-4862
ISBN 0-471-86681-4 AACR2

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

To my wife Helen

Preface

Algebra is a mathematical shorthand for language, and matrices are a shorthand for algebra. Consequently, a special value of matrices is that they enable many mathematical operations, especially those arising in statistics and the quantitative sciences, to be expressed concisely and with clarity. The algebra of matrices is, of course, in no way new, but its presentation is often so surrounded by the trappings of mathematical generality that assimilation can be difficult for readers who have only limited ability or training in mathematics. Yet many such people nowadays find a knowledge of matrix algebra necessary for their work, especially where statistics and/or computers are involved. It is to these people that I address this book; and for them, I have attempted to keep the mathematical presentation as informal as possible.

The pursuit of knowledge frequently involves collecting data; and those responsible for the collecting must appreciate the need for analyzing their data to recover and interpret the information contained therein. Such people must therefore understand some of the mathematical tools necessary for this analysis, to an extent either that they can carry out their own analysis, or that they can converse with statisticians and mathematicians whose help will otherwise be needed. One of the necessary tools is matrix algebra. It is becoming as necessary to science today as elementary calculus has been for generations. Matrices originated in mathematics more than a century ago, but their broad adaptation to science is relatively recent, prompted by the widespread acceptance of statistical analysis of data, and of computers to do that analysis; both statistics and computing rely heavily on matrix algebra. The purpose of this book is therefore that of bringing to a broad spectrum of readers a knowledge of matrix algebra that is useful in the statistical analysis of data and in statistics generally.

The basic prerequisite for using the book is high school algebra. Differential calculus is used on only a few pages, which can easily be omitted;

nothing will be lost insofar as a general understanding of matrix algebra is concerned. Proofs and demonstrations of most of the theory are given, for without them the presentation would be lifeless. But in every chapter the theoretical development is profusely illustrated with elementary numerical examples and with illustrations taken from a variety of applied sciences. And the last three chapters are devoted solely to uses of matrix algebra in statistics, with Chapters 14 and 15 outlining two of the most widely used statistical techniques: regression and linear models.

The mainstream of the book is its first eleven chapters, beginning with one on introductory concepts that includes a discussion of subscript and summation notation. This is followed by four chapters dealing with basic arithmetic, special matrices, determinants and inverses. Chapters 6 and 7 are on rank and canonical forms, 8 and 9 deal with generalized inverses and solving linear equations, 10 is a collection of results on partitioned matrices and 11 describes eigenvalues and eigenvectors. Background theory for Chapter 11 is collected in an appendix, Chapter 11A, some summaries and miscellaneous topics make up Chapter 12, statistical illustrations constitute Chapter 13, and Chapters 14 and 15 describe regression and linear models. All chapters except the last two end with exercises.

Occasional sections and paragraphs can be omitted at a first reading, especially by those whose experience in mathematics is somewhat limited. These portions of the book are printed in small type and, generally speaking, contain material subsidiary to the main flow of the text—material that may be a little more advanced in mathematical presentation than the general level otherwise maintained.

Chapters, and sections within chapters, are numbered with Arabic numerals 1, 2, 3,. . . . Within-chapter references to sections are by section number, but references across chapters use the decimal system; e.g., Section 1.3 is Section 3 of Chapter 1. These numbers are also shown in the running head of each page; e.g., [1.3] is found on page 4. Numbered equations are (1), (2),. . . , within each chapter. Those of one chapter are seldom referred to in another, but when they are, the chapter reference is explicit; otherwise "equation (3)," or more simply "(3)," means the equation numbered (3) in the chapter concerned. Exercises are in un-numbered sections and are referenced by their chapter number; e.g., Exercise 6.2 is Exercise 2 at the end of Chapter 6.

I am greatly indebted to George P. H. Styan for his exquisitely thorough readings of two drafts of the manuscript and his extensive and very helpful array of comments. Harold V. Henderson's numerous suggestions for the final manuscript were equally as helpful. Readers of *Matrix Algebra for the*

Biological Sciences (Wiley, 1966), and students in fifteen years of my matrix algebra course at Cornell have also contributed many useful ideas. Particular thanks go to Mrs. Helen Seamon for her superb accuracy on the typewriter, her patience and fantastic attention to detail; such attributes are greatly appreciated.

Ithaca, New York
May 1982

SHAYLE R. SEARLE

Contents

5. Inverse Matrices (*Continued*)