Graduate Texts in Mathematics

Donald W. Barnes
John M. Mack

An Algebraic Introduction to Mathematical Logic

Springer—Verlag
New York Heidelberg Berlin
World Publishing Corporation Beijing China

An Algebraic Introduction to Mathematical Logic

Springer-Verlag New York Heidelberg Berlin
World Publishing Corporation, Beijing, China

Donald W. Barnes
John M. Mack
The University of Sydney
Department of Pure Mathematics
Sydney, N.S.W. 2006
Australia

Reprinted in China by World Publishing Corporation

For distribution and sale in the People's Republic of China only

只限在中华人民共和国发行

AMS Subject Classifications
Primary: 0201
Secondary: 02B05, 02B10, 02F15, 02G05, 02G10, 02G15, 02G20, 02H05, 02H13, 02H15, 02H20, 02H25

Library of Congress Cataloging in Publication Data

Barnes, Donald W
An algebraic introduction to mathematical logic.
(Graduate texts in mathematics; v. 22)
Bibliography: p. 115
Includes index.
1. Logic, Symbolic and mathematical. 2. Algebraic logic. I. Mack, J. M., joint author. II. Title.
III. Series.
QA9.B27 511'.3 74-22241

Printed in the United States of America.

ISBN 0-387-90109-4 Springer-Verlag New York Heidelberg Berlin
ISBN 3-540-90109-4 Springer-Verlag Berlin Heidelberg New York
ISBN 7-5062-0052-X World Publishing Corporation China

Preface

This book is intended for mathematicians. Its origins lie in a course of lectures given by an algebraist to a class which had just completed a substantial course on abstract algebra. Consequently, our treatment of the subject is algebraic. Although we assume a reasonable level of sophistication in algebra, the text requires little more than the basic notions of group, ring, module, etc. A more detailed knowledge of algebra is required for some of the exercises. We also assume a familiarity with the main ideas of set theory, including cardinal numbers and Zorn's Lemma.

In this book, we carry out a mathematical study of the logic used in mathematics. We do this by constructing a mathematical model of logic and applying mathematics to analyse the properties of the model. We therefore regard all our existing knowledge of mathematics as being applicable to the analysis of the model, and in particular we accept set theory as part of the meta-language. We are not attempting to construct a foundation on which all mathematics is to be based—rather, any conclusions to be drawn about the foundations of mathematics come only by analogy with the model, and are to be regarded in much the same way as the conclusions drawn from any scientific theory.

The construction of our model is greatly simplified by our using universal algebra in a way which enables us to dispense with the usual discussion of essentially notational questions about well-formed formulae. All questions and constructions relating to the set of well-formed formulae are handled by our Theorems 2.2 and 4.3 of Chapter I. Our use of universal algebra also provides us with a convenient method for discussing free variables (and avoiding reference to bound variables), and it also permits a simple neat statement of the Substitution Theorem (Theorems 4.11 of Chapter II and 4.3 of Chapter IV).

Chapter I develops the necessary amount of universal algebra. Chapters II and III respectively construct and analyse a model of the Propositional Calculus, introducing in simple form many of the ideas needed for the more complex First-Order Predicate Calculus, which is studied in Chapter IV. In Chapter V, we consider first-order mathematical theories, i.e., theories built on the First-Order Predicate Calculus, thus building models of parts of mathematics. As set theory is usually regarded as the basis on which the rest of mathematics is constructed, we devote Chapter VI to a study of first-order Zermelo-Fraenkel Set Theory. Chapter VII, on Ultraproducts, discusses a technique for constructing new models of a theory from a given collection of models. Chapter VIII, which is an introduction to Non-Standard Analysis, is included as an example of mathematical logic assisting in the study of another branch of mathematics. Decision processes are investigated in Chapter IX, and we prove there the non-existence of decision processes for a number of problems. In Chapter X, we discuss two decision problems from other branches of mathematics and indicate how the results of Chapter IX may be applied.

This book is intended to make mathematical logic available to mathematicians working in other branches of mathematics. We have included what we consider to be the essential basic theory, some useful techniques, and some indications of ways in which the theory might be of use in other branches of mathematics.

We have included a number of exercises. Some of these fill in minor gaps in our exposition of the section in which they appear. Others indicate aspects of the subject which have been ignored in the text. Some are to help in understanding the text by applying ideas and methods to special cases. Occasionally, an exercise asks for the construction of a FORTRAN program. In such cases, the solution should be based on integer arithmetic, and not depend on any special logical properties of FORTRAN or of any other programming language.

The layout of the text is as follows. Each chapter is divided into numbered sections, and definitions, theorems, exercises, etc. are numbered consecutively within each section. For example, the number 2.4 refers to the fourth item in the second section of the current chapter. A reference to an item in some other chapter always includes the chapter number in addition to item and section numbers.

We thank the many mathematical colleagues, particularly Paul Halmos and Peter Hilton, who encouraged and advised us in this project. We are especially indebted to Gordon Monro for suggesting many improvements and for providing many exercises. We thank Mrs. Blakestone and Miss Kicinski for the excellent typescript they produced.

Donald W. Barnes, John M. Mack

Table of Contents

Chapter I

Universal Algebra

§1 Introduction

The reader will be familiar with the presentation and study of various algebraic systems (for example, groups, rings, modules) as axiomatic systems consisting of sets with certain operations satisfying certain conditions. The reader will also be aware that ideas and theorems, useful for the study of one type of system, can frequently be adapted to other related systems by making the obvious necessary modifications.

In this book we shall study and use a number of systems whose types are related, but which are possibly unfamiliar to the reader. Hence there is obvious advantage in beginning with the study of a single axiomatic theory which includes as special cases all the systems we shall use. This theory is known as universal algebra, and it deals with systems having arbitrary sets of operations. We shall want to avoid, as far as possible, axioms asserting the existence of elements with special properties (for example, the identity element in group theory), preferring the axioms satisfied by operations to take the form of equations, and we shall be able to achieve this by giving a sufficiently broad definition of "operation". We first recall some elementary facts.

An n-ary relation ρ on the sets $A_1, \ldots, A_n$ is specified by giving those ordered n-tuples $(a_1, \ldots, a_n)$ of elements $a_i \in A_i$ which are in the relation ρ. Thus such a relation is specified by giving those elements $(a_1, \ldots, a_n)$ of the product set $A_1 \times \cdots \times A_n$ which are in ρ, and hence an n-ary relation on $A_1, \ldots, A_n$ is simply a subset of $A_1 \times \cdots \times A_n$. For binary relations, the notation "$a_1 \rho a_2$" is commonly used to express "(a_1, a_2) is in the relation ρ", but we shall usually write this as either "$(a_1, a_2) \in \rho$" or "$\rho(a_1, a_2)$", because each of these notations extends naturally to n-ary relations for any n.

A function $f: A \to B$ is a binary relation on A and B such that, for each $a \in A$, there is exactly one $b \in B$ for which $(a, b) \in f$. It is usual to write this as $f(a) = b$. A function $f(x, y)$ "of two variables" $x \in A, y \in B$, with values in C, is simply a function $f: A \times B \to C$. For each $a \in A$ and $b \in B$, $(a, b) \in A \times B$ and $f((a, b)) \in C$. It is of course usual to omit one set of brackets. There are advantages in retaining the variables x, y in the function notation. Later in this chapter, we will discuss what is meant by variables and give a definition which will justify their use.

Preliminary Definition of Operation. *An n-ary operation on the set A is a function $t: A^n \to A$. The number n is called the arity of t.*

Examples

1.1. Multiplication in a group is a binary operation. The $*$-product of two elements a, b is written $a*b$ or simply ab instead of the more systematic $*(a, b)$.

1.2. In a group G, we can define a unary operation $i: G \to G$ by putting $i(a) = a^{-1}$.

1.3. A 0-ary operation on a set A is a function from the set A^0 (whose only element is the empty set $\varnothing$) to the set A, and hence can be regarded as a distinguished element of A. Such an operation arises naturally in group theory, where the 0-ary operation e gives the identity element of the group G.

One often considers several different groups in group theory. If G, H are groups, each has its multiplication operation: $*_G: G \times G \to G$ and $*_H: H \times H \to H$, but one rarely uses distinctive notations for the two multiplications. In practice, the same notation $*$ is used for both, and in fact multiplication is regarded as an operation defined for all groups. The definition of operation given above is clearly not adequate for this usage of the word.

Here is another example demonstrating that our preliminary definition of operation does not match common usage. A ring R is usually defined as a set R with two binary operations $+$, $\times$ satisfying certain axioms. A commonly occurring example of a ring is the zero ring where $R = \{0\}$. In this case, there is only one function $R \times R \to R$, and so $+$, $\times$ are the same function, even though $+$ and $\times$ are still considered distinct operations.

We now give a series of definitions which will overcome the objections raised above.

Definition 1.4. A *type* $\mathscr{T}$ is a set T together with a function $\text{ar}: T \to \mathbf{N}$, from T into the non-negative integers. We shall write $\mathscr{T} = (T, \text{ar})$, or, more simply, abuse notation and denote the type by T. It is also convenient to denote by T_n the set $\{t \in T \mid \text{ar}(t) = n\}$.

Definition 1.5. An *algebra A of type T*, or a *T-algebra*, is a set A together with, for each $t \in T$, a function $t_A: A^{\text{ar}(t)} \to A$. The elements $t \in T_n$ are called n-ary T-algebra operations.

Observe that each t_A is an operation on the set A in the sense of our preliminary definition of operation. As is usual, we shall write simply $t(a_1, \ldots, a_n)$ for the element $t_A(a_1, \ldots, a_n)$, and we shall denote the algebra by the same symbol A as is used to denote its set of elements.

Examples

1.6. Rings may be considered as algebras of type $T = (\{0, -, +, \cdot\}, \text{ar})$, where $\text{ar}(0) = 0$, $\text{ar}(-) = 1$, $\text{ar}(+) = 2$, $\text{ar}(\cdot) = 2$. We do not claim that such T-algebras are necessarily rings, we simply assert that each ring is an example of a T-algebra for the T given above.

1.7. If R is a given ring, then a module over R may be regarded as a particular example of a T-algebra of type $T = (\{0, -, +\} \cup R, \mathrm{ar})$, where $\mathrm{ar}(0) = 0$, $\mathrm{ar}(-) = 1$, $\mathrm{ar}(+) = 2$, and $\mathrm{ar}(\lambda) = 1$ for each $\lambda \in R$. The first three operations specify the group structure of the module, while the remaining operations correspond to the action of the ring elements.

1.8. Let S be a given ring. Rings R which contain S as subring may be considered as T-algebras, where $T = (\{0, -, +, \cdot\} \cup S, \mathrm{ar})$, $\mathrm{ar}(0) = 0$, $\mathrm{ar}(-) = 1$, $\mathrm{ar}(+) = 2$, $\mathrm{ar}(\cdot) = 2$, and $\mathrm{ar}(s) = 0$ for each $s \in S$. The effect of the S-operations is to distinguish certain elements of R.

Definition 1.9. T-algebras A, B are *equal* if and only if $A = B$ and $t_A = t_B$ for all $t \in T$.

Exercise 1.10. Give an example of unequal T-algebras on the same set A.

Definition 1.11. If A is a T-algebra, a subset B of A is called a *T-subalgebra* of A if it forms a T-algebra with operations the restrictions to B of those on A, i.e., if for all n and for all $t \in T_n$ and $b_1, \ldots, b_n \in B$, we have $t_A(b_1, \ldots, b_n) \in B$.

Any intersection of subalgebras is a subalgebra, and so, given any subset X of A, there is a unique smallest subalgebra containing X—namely, the subalgebra $\cap\{U | U \text{ subalgebra of } A, U \supseteq X\}$. We call this the subalgebra generated by X and denote it by $\langle X \rangle_T$, or if there is no risk of confusion, by $\langle X \rangle$.

Exercises

1.12. A is a T-algebra. Show that $\varnothing$ is a subalgebra if and only if $T_0 = \varnothing$. Show that for all T, every T-algebra has a unique smallest subalgebra.

Many familiar algebraic systems may be regarded as T-algebras for more than one choice of T. However, the subsets which form T-subalgebras may well depend on the choice of T.

1.13. Groups may be regarded as special cases of T-algebras where $T = (\{*\}, \mathrm{ar})$ with $\mathrm{ar}(*) = 2$, or of T'-algebras, where $T' = (\{e, i, *\}, \mathrm{ar})$, $\mathrm{ar}(e) = 0$, $\mathrm{ar}(i) = 1$, $\mathrm{ar}(*) = 2$. Show that every T'-subalgebra of a group is a subgroup, but that not every non-empty T-subalgebra need be a group. Show that if G is a finite group, then every non-empty T-subalgebra of G is itself a group.

Definition 1.14. Let A, B be T-algebras. A *homomorphism* of A into B is a function $\varphi: A \to B$ such that, for all $t \in T$ and all $a_1, \ldots, a_n \in A$ ($n = \mathrm{ar}(t)$), we have

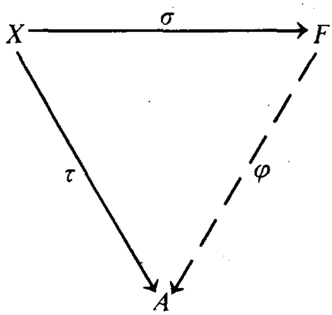
$$\varphi(t_A(a_1, \ldots, a_n)) = t_B(\varphi(a_1), \ldots, \varphi(a_n)).$$

This condition is often expressed as "*φ preserves all the operations of T*".

Clearly, the composition of two homomorphisms is a homomorphism. Further, if $\varphi: A \to B$ is a homomorphism and is invertible, then the inverse function $\varphi^{-1}: B \to A$ is also a homomorphism. In this case we call φ an *isomorphism* and say that A and B are *isomorphic*.

§2 Free Algebras

Definition 2.1. Let X be any set, let F be a T-algebra and let $\sigma: X \to F$ be a function. We say that F (more strictly (F, σ)) is a *free T-algebra* on the set X of *free generators* if, for every T-algebra A and function $\tau: X \to A$, there exists a unique homomorphism $\varphi: F \to A$ such that $\varphi\sigma = \tau$:



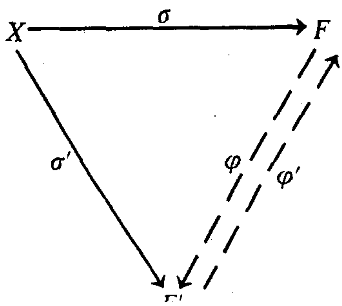
Observe that if (F, σ) is free, then σ is injective. For it is easily seen that there exists a T-algebra with more than one element, and hence if x_1, x_2 are distinct elements of X, then for some A and τ we have $\tau(x_1) \neq \tau(x_2)$, which implies $\sigma(x_1) \neq \sigma(x_2)$.

The next theorem asserts the existence of a free T-algebra on a set X, and the proof is constructive. Informally, one could describe the free T-algebra on X as the collection of all formal expressions that can be formed from X and T by using only finitely many elements of X and T in any one expression. But to say precisely what is meant by a formal expression in the elements of X using the operations of T is tantamount to constructing the free algebra.

Theorem 2.2. *For any set X and any type T, there exists a free T-algebra on X. This free T-algebra on X is unique up to isomorphism.*

Proof. (a) *Uniqueness.* We show first that if (F, σ) is free on X, and if $\varphi: F \to F$ is a homomorphism such that $\varphi\sigma = \sigma$, then $\varphi = 1_F$, the identity map on F. To show this, we take $A = F$ and $\tau = \sigma$ in the defining condition. Then $1_F: F \to F$ has the required property for φ, and hence by its uniqueness is the only such map.

Now let (F, σ) and (F', σ') be free on X.



Since (F, σ) is free, there exists a homomorphism $\varphi: F \to F'$ such that $\varphi\sigma = \sigma'$. Since (F', σ') is free, there exists a homomorphism $\varphi': F' \to F$ such that $\varphi'\sigma' = \sigma$. Hence $\varphi'\varphi\sigma = \varphi'\sigma' = \sigma$, and by the result above, $\varphi'\varphi = 1_F$. Similarly, $\varphi\varphi' = 1_{F'}$. Thus φ, φ' are mutually inverse isomorphisms, and so uniqueness is proved.

(b) *Existence.* An algebra F will be constructed as a union of sets F_n ($n \in \mathbb{N}$), which are defined inductively as follows.

(i) F_0 is the disjoint union of X and T_0.

(ii) Assume F_r is defined for $0 \leqslant r < n$. Then define

$$F_n = \left\{(t, a_1, \ldots, a_k) \middle| t \in T, \operatorname{ar}(t) = k, a_i \in F_{r_i}, \sum_{i=1}^{k} r_i = n - 1\right\}.$$

(iii) Put $F = \bigcup_{n \in \mathbb{N}} F_n$.

The set F is now given. To make it into a T-algebra, we must specify the action of the operations $t \in T$.

(iv) If $t \in T_k$ and $a_1, \ldots, a_k \in F$, put $t(a_1, \ldots, a_k) = (t, a_1, \ldots, a_k)$. In particular, if $t \in T_0$, then t_F is the element t of F_0.

This makes F into a T-algebra. To complete the construction, we must give the map $\sigma: X \to F$.

(v) For each $x \in X$, put $\sigma(x) = x \in F_0$.

Finally, we have to prove that F is free on X, i.e., we must show that if A is any T-algebra and $\tau: X \to A$ any map of X into A, then there exists a unique homomorphism $\varphi: F \to A$ such that $\varphi\sigma = \tau$. We do this by constructing inductively the restriction φ_n of φ to F_n and by showing that φ_n is completely determined by τ and the φ_k for $k < n$.

We have $F_0 = T_0 \cup X$. The homomorphism condition requires $\varphi_0(t_F) = t_A$ for $t \in T_0$, while for $x \in X$ we require $\varphi\sigma(x) = \tau(x)$, and so we must have

$\varphi_0(x) = \tau(x)$. Thus $\varphi_0: F_0 \to A$ is defined, and is uniquely determined by the conditions to be satisfied by φ.

Suppose that φ_k is defined and uniquely determined for $k < n$. An element of F_n $(n > 0)$ is of the form $(t, a_1 \ldots, a_k)$, where $t \in T_k$, $a_i \in F_{r_i}$ and $\sum_{i=1}^{k} r_i = n - 1$. Thus $\varphi_{r_i}(a_i)$ is already uniquely defined for $i = 1, \ldots, k$. Furthermore, since $(t, a_1, \ldots, a_k) = t(a_1, \ldots, a_k)$, and since the homomorphism property of φ requires that

$$\varphi(t, a_1, \ldots, a_k) = t(\varphi(a_1), \ldots, \varphi(a_k)),$$

we must define

$$\varphi_n(t, a_1, \ldots, a_k) = t(\varphi_{r_1}(a_1), \ldots, \varphi_{r_k}(a_k)).$$

This determines φ_n uniquely, and as each element of F belongs to exactly one subset F_n, on putting $\varphi(\alpha) = \varphi_n(\alpha)$ for $\alpha \in F_n$ $(n \geqslant 0)$, we see that φ is a homomorphism from F to A satisfying $\varphi\sigma(x) = \varphi_0(x) = \tau(x)$ for all $x \in X$ as required, and that φ is the only such homomorphism. □

The above inductive construction of the free T-algebra F fits in with its informal description—each F_n is a collection of "T-expressions", increasing in complexity with n. The notion of a T-expression is useful for an arbitrary T-algebra, so we shall formalise it, making use of free T-algebras to do so.

Let A be any T-algebra, and let F be the free T-algebra on the set $X_n = \{x_1, \ldots, x_n\}$. For any (not necessarily distinct) elements $a_1, \ldots, a_n \in A$, there exists a unique homomorphism $\varphi: F \to A$ with $\varphi(x_i) = a_i$ $(i = 1, \ldots, n)$. If $w \in F$, then $\varphi(w)$ is an element of A which is uniquely determined by $a_1, \ldots, a_n$. Hence we may define a function $w_A: A^n \to A$ by putting $w_A(a_1, \ldots, a_n) = \varphi(w)$. We omit the subscript A and write simply $w(a_1, \ldots, a_n)$. If in particular we take $A = F$ and $a_i = x_i$ $(i = 1, \ldots, n)$, then φ is the identity and $w(x_1, \ldots, x_n) = w$.

Definition 2.3. A *T-word* in the *variables* $x_1, \ldots, x_n$ is an element of the free T-algebra on the set $X_n = \{x_1, \ldots, x_n\}$ of free generators.

Definition 2.4. A *word* in the elements $a_1, \ldots, a_n$ of a T-algebra A is an element $w(a_1, \ldots, a_n) \in A$, where w is a T-word in the variables $x_1, \ldots, x_n$.

We have used and even implicitly defined the term "variable" in the above definitions. In normal usage, a variable is "defined" as a symbol for which any element of the appropriate kind may be substituted. We give a formal definition of variable, confirming that our variables have this usual property.

Definition 2.5. A *T-algebra variable* is an element of the free generating set of a free T-algebra.

Among the words in the variables $x_1, \ldots, x_n$ are the words x_i $(i = 1, \ldots, n)$, having the property that $x_i(a_1, \ldots, a_n) = a_i$. Thus variables may also be regarded as coordinate functions. The concept of a coordinate function certainly provides the most convenient definition of variable for use in analysis. For example, when we speak of a function $f(x, y)$ as a function of two real variables x, y, we have a function f, defined on some subset of $\mathbf{R} \times \mathbf{R}$, together with coordinate projections $x(a, b) = a$, $y(a, b) = b$ $(a, b \in \mathbf{R})$, and $f(x, y)$ is in fact the composite function $f(a, b) = f(x(a, b), y(a, b))$.

Exercises

2.6. T consists of one unary operation, and F is the free T-algebra on a one-element set X. How many elements are there in F_n? How many elements are there in F?

2.7. If T is empty and X is any set, show that X is the free T-algebra on X.

2.8. T consists of a single binary operation, and F is the free T-algebra on a one-element set X. How many elements are there in F?

2.9. If T consists of one 0-ary operation and one 2-ary operation, and if $X = \varnothing$, then the free T-algebra F on X is countable.

2.10. T is finite or countable, and contains at least one 0-ary operation and at least one operation t with $\mathrm{ar}(t) > 0$. X is finite or countable. Prove that F is countable.

§3 Varieties of Algebras

Let F be the free T-algebra on the countable set $X = \{x_1, x_2, \ldots\}$ of variables. Although each element of F is a word in some finite subset $X_n = \{x_1, \ldots, x_n\}$, we shall consider sets of words for which there may be no bound to the number of variables in the words.

Definition 3.1. An *identical relation* on T-algebras is a pair (u, v) of elements of F.

There is an n for which u, v are in the free algebra on X_n, and we say that (u, v) is an *n-variable identical relation* for any such n.

Definition 3.2. The T-algebra A *satisfies* the n-variable identical relation (u, v), or (u, v) is a *law* of A, if $u(a_1, \ldots, a_n) = v(a_1, \ldots, a_n)$ for all $a_1, \ldots, a_n \in A$.

Equivalently, (u, v) is a law of A if $\varphi(u) = \varphi(v)$ for every homomorphism $\varphi: F \to A$.

Definition 3.3. Let L be a set of identical relations on T-algebras. The class V of all T-algebras which satisfy all the identical relations in L is called the *variety of T-algebras defined by L*. The *laws of the variety* are all the identical relations satisfied by every algebra of V.

Note that the set of laws of the variety includes L, but may be larger.

Examples

3.4. T consists of a single binary operation $*$, and L has the one element $(x_1*(x_2*x_3), (x_1*x_2)*x_3)$. If A satisfies this identical relation, then $a*(b*c) = (a*b)*c$ for all $a, b, c \in A$. Thus the operation on A is associative and A is a semigroup. The variety defined by L in this case is the class of all semigroups.

3.5. T consists of 0-ary, 1-ary and 2-ary operations e, i, $*$ respectively. L has the three elements

$$(x_1*(x_2*x_3), (x_1*x_2)*x_3),$$
$$(e*x_1, x_1),$$
$$(i(x_1)*x_1, e).$$

The first law ensures that $*$ is an associative operation in every algebra of the variety defined by L. The second shows that the distinguished element e is always a left identity, while the third guarantees that $i(a)$ is a left inverse of the element a. Hence the algebras of the variety are groups.

Exercises

3.6. Show that the class of all abelian groups is a variety.

3.7. R is a ring with 1. Show that the class of unital left R-modules is a variety.

3.8. S is a commutative ring with 1. Show that the class of commutative rings R with $1_R = 1_S$ and which contain S as a subring is a variety.

3.9. Is the class of finite groups a variety?

§4 Relatively Free Algebras

Let V be the variety of T-algebras defined by the set L of laws.

Definition 4.1. A T-algebra R in the variety V is the *(relatively) free algebra of V* on the set X of *(relatively) free generators* (where a function $\sigma: X \to R$ is given, usually as an inclusion) if, for every algebra A in V and every function $\tau: X \to A$, there exists a unique homomorphism $\varphi: R \to A$ such that $\varphi\sigma = \tau$.

This definition differs from the earlier definition of a free algebra only in that we consider here only algebras in V.

Definition 4.2. An algebra is *relatively free* if it is a free algebra of some variety.

Theorem 4.3. *For any type T, and any set L of laws, let V be the variety of T-algebras defined by L. For any set X, there exists a free T-algebra of V on X.*

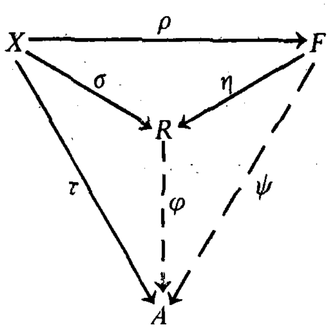
Proof: Let (F, ρ) be the free T-algebra on X. A congruence relation on F is defined by putting $u \sim v$ (where $u, v \in F$) if $\varphi(u) = \varphi(v)$ for every homomorphism φ of F into an algebra in V. Clearly $\sim$ is an equivalence relation on F. If now $t \in T_k$ and $u_i \sim v_i$ $(i = 1, \ldots, k)$, then for every such homomorphism φ, $\varphi(u_i) = \varphi(v_i)$, and so

$$\varphi(t(u_1, \ldots, u_k)) = t(\varphi(u_1), \ldots, \varphi(u_k)) = t(\varphi(v_1), \ldots, \varphi(v_k)) = \varphi(t(v_1, \ldots, v_k)),$$

verifying that φ is a congruence relation.

We define R to be the set of congruence classes of elements of F with respect to this congruence relation. Denoting the congruence class containing u by $\bar{u}$, we define the action of $t \in T_k$ on R by putting $t(\bar{u}_1, \ldots, \bar{u}_k) = \overline{t(u_1, \ldots, u_k)}$. This definition is independent of the choice of representatives $u_1, \ldots, u_k$ of the classes $\bar{u}_1, \ldots, \bar{u}_k$, and makes R a T-algebra. Also, the map $u \to \bar{u}$ is clearly a homomorphism $\eta: F \to R$. Finally, we define $\sigma: X \to R$ by $\sigma(x) = \overline{\rho(x)}$.

We now prove that (R, σ) is relatively free on X. Let A be any algebra in V, and let $\tau: X \to A$ be any function from X into A. Because (F, ρ) is free, there exists a unique homomorphism $\psi: F \to A$ such that $\psi\rho = \tau$.



For $\bar{u} \in R$, we define $\varphi(\bar{u}) = \psi(u)$. This is independent of the choice of representative u of the element $\bar{u}$, since if $\bar{u} = \bar{v}$, then $\psi(u) = \psi(v)$. The map $\varphi: R \to A$ is clearly a homomorphism, and $\varphi\sigma = \varphi\eta\rho = \psi\rho = \tau$. If $\varphi': R \to A$ is another homomorphism such that $\varphi'\sigma = \tau$, then $\varphi'\eta\rho = \tau$ and therefore $\varphi'\eta = \psi$. Consequently for each element $\bar{u} \in R$ we have

$$\varphi'(\bar{u}) = \varphi'\eta(u) = \psi(u) = \varphi(\bar{u}),$$

and hence $\varphi' = \varphi$. $\square$

When considering only the algebras of a given variety V, we may redefine variables and words accordingly. Thus we define a V-*variable* as an element of the free generating set of a free algebra of V, and a V-*word* in the V-variables $x_1, \ldots, x_n$ as an element of the free algebra of V on the free generators $\{x_1, \ldots, x_n\}$.

Examples

4.4. T consists of a single binary operation which we shall write as juxtaposition. Let V be the variety of associative T-algebras. Then all products in the free T-algebra obtained by any bracketing of $x_1, \ldots, x_n$, taken in that order, are congruent under the congruence relation used in our construction of the relatively free algebra, and correspond to the one word $x_1x_2 \cdots x_n$ of V. We observe that in this example, all elements of the absolutely free algebra F, which map to a given element $x_1x_2 \cdots x_n$ of the relatively free algebra, come from the same layer F_{n-1} of F.

4.5. T consists of a 0-ary, a 1-ary and a 2-ary operation. V is the variety of abelian groups, defined by the laws given in Example 3.5 together with the law (x_1x_2, x_2x_1). In this case, the relatively free algebra on $\{x_1, \ldots, x_n\}$ is the set of all $x_1^{r_1}x_2^{r_2} \cdots x_n^{r_n}$ (or equivalently the set of all n-tuples $(r_1, \ldots, r_n)$) with $r_i \in \mathbf{Z}$. Here the layer property of Example 4.4 does not hold, because, for example, we have the identity $e \in F_0$, $x_1^{-1} \in F_1$, $x_1^{-1}*x_1 \in F_2$ and yet $\overline{e} = \overline{x_1^{-1}*x_1}$.

Exercises

4.6. K is a field. Show that vector spaces over K form a variety V of algebras, and that every vector space over K is a free algebra of V.

4.7. R is a commutative ring with 1 and V is the variety of commutative rings S which contain R as a subring and in which 1_R is a multiplicative identity of S. Show that the free algebra of V on the set X of variables is the polynomial ring over R in the elements of X.