Johnson and Kiokemeister

CALCULUS

with analytic geometry

SECOND EDITION

with analytic geometry

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ALLYN and BACON, Inc. Boston, 1960

First printing . . . . January, 1960
Second printing . . . . June, 1960

Library of Congress Catalog Card Number: 59-10051

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Preface

THIS SECOND EDITION differs from the first in that some of the chapters have been improved and modified in the light of experience, a new chapter on three-dimensional vector analysis has been added, and supplementary exercises have been included.

Chapter 6 as now presented stresses graphical methods for determining extrema of a function. Rolle's theorem and the mean value theorem are introduced here (rather than in Chapter 14 as in the first edition). This is followed up by a section on intermediate value theorems in Chapter 8, on the definite integral. An intuitive discussion of the derivative of exponential functions is now included in Chapter 10. The section on two-dimensional vector algebra in Chapter 15 has been rewritten from an algebraic standpoint. This leads naturally to the three-dimensional vector analysis of Chapter 18. A recent elementary proof of the fundamental theorem of algebra has been added to Chapter 19, on partial differentiation.

The appendix has been enlarged to include a list of formulas from trigonometry and tables of exponential, logarithmic, and trigonometric functions. Additional exercises for each of the chapters are included in the set of supplementary exercises.

It is our conviction that college students, particularly in the first two years, not only are able to understand and appreciate rigorous mathematical theory but also are more interested in courses containing both rigorous proofs and applications of the theorems. This conviction led us to write a book containing a theoretical basis of the calculus together with applications of the methods and results. An early start in theoretical mathematics has an added advantage in that it allows a wide selection of meaningful courses in the last two years of college for those students continuing in mathematics. Equally important, students taking only two years of mathematics will get an insight into the methods as well as the applications of the calculus.

Although this is primarily a calculus book, enough analytic geometry has been included to make it self-contained in that respect. For students who have had analytic geometry, Chapter 2 on the line and the part of Chapter 7 on conic sections may be omitted without affecting the continuity of the book. The book has been designed to be used by students having good training in algebra, plane geometry, and trigonometry.

Every effort has been made to give an intuitive discussion of each new concept prior to its rigorous development. The more difficult theoretical parts have been put in separate sections, and may be omitted at the discretion of the instructor. Following the theory, the natural geometrical and physical applications have been presented.

Some of the other unusual features of the book follow:

In Chapter 4, limits of the linear, quadratic, reciprocal, and square-root functions are discussed before the statement of the limit theorems. Then the limit theorems are shown to follow from these special limits and a general composite limit theorem.

A particularly complete treatment of extrema of a function is given in Chapter 6. A novel feature of this chapter is the emphasis on graphical means of determining extrema.

The completeness property of the real number system is stated in terms of least upper bounds and greatest lower bounds in Chapter 8, and then is used in the definition of lower and upper integrals of a continuous function. The equality of the lower and upper integrals is proved on the assumption that a function continuous in a closed interval has a simply stated uniform continuity property in that interval. This proof may be replaced by a geometrical argument if the instructor feels that it is too complicated for his class. A similar treatment of the double integral is to be found in Chapter 20.

Sequences are introduced in Chapter 9 for the specific purpose of showing that the integral equals the limit of a sequence of Riemann sums. The sigma and delta notations are used for the first time in this chapter.

A full account of the differential calculus of the elementary transcendental functions is given in Chapter 10. Special sections on inverse functions, exponential laws of growth and decay, and the definition of partial derivatives are also included.

The formal integration of Chapters 11 and 12 contains a section on separable differential equations. Featured in the latter chapter is a general substitution formula that validates trigonometric and related substitutions for evaluating integrals.

A transition from the elementary topics of differentiation and integration to some of the deeper properties of functions occurs in Chapter 14. Here the boundedness properties of a continuous function are proved, and the mean value theorem is brought into play in the discussion of indeterminate forms, improper integrals, and the finite Taylor's theory.

Vectors are introduced in Chapter 15 and studied further in Chapter 18, where three-dimensional vector analysis is applied to geometric and physical problems. Solid analytic geometry is studied first in Chapter 17 without vectors, and then again in Chapter 18 with the aid of vectors.

A thorough treatment of elementary convergence theory of infinite series is given in Chapter 16. Least upper bounds are again used in defining the radius of convergence of a power series. Proofs are given that a power series may be differentiated and integrated termwise within its interval of convergence.

Chapters 19 and 20 are given over to a brief introduction to the calculus of functions of several variables.

The final chapter on differential equations features a complete discussion of second-order linear differential equations with constant coefficients.

We have attempted to make the chapters as independent of each other as possible so that the book might be used for a second calculus course following a variety of different first courses. Essentially all of the material in this book has been tested in the classroom by the authors and their colleagues and has been improved as a result of this experience. The publishers have also had many critical readings of the book by other teachers. To these and the many other people who have helped so greatly in the formation of this book we express our deep appreciation.

R.E.J.

F.L.K.

Contents

1

Topics from algebra

CALCULUS, LIKE ALGEBRA, is concerned with properties of numbers. However, unlike algebra, it has to do primarily with the concept of limit rather than with such purely algebraic concepts as factorization, solving equations, etc. Limits will be defined in the fourth chapter. The present chapter will be devoted to topics that are perhaps of secondary importance in algebra but are of primary importance in the calculus.

1. Numbers

The number system used in elementary calculus is the system of real numbers; in more advanced calculus, the complex number system proves to be very useful. Since we are concerned with elementary calculus in this book, we shall limit our discussion almost exclusively to real numbers.

We do not intend to give a detailed account of the real number system at this point. Rather, we shall briefly describe the different types of numbers that make up the real number system. Deeper aspects of the real number system will be brought in later as they are needed.

Included among the real numbers are the *integers*

$$\cdots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \cdots$$

and the ratios of integers, called *rational numbers.* Thus each rational number has the form p/q, where p and q are integers with $q \neq 0$.

For example, $\frac{2}{3}$, $-\frac{7}{8}$, $5 = \frac{5}{1}$, and $1.23 = \frac{123}{100}$ are rational numbers. Every integer p is also a rational number $p/1$.

There are real numbers that are not rational numbers; such numbers are called *irrational numbers.* For example, $\sqrt[3]{2}$, $1/(\sqrt{5} + 1)$, and π are irrational numbers. As another example, the quadratic equation

$$x^2 - 2x - 2 = 0$$

has as its solutions the irrational numbers

$$1 + \sqrt{3} \quad \text{and} \quad 1 - \sqrt{3}.$$

Complex numbers are numbers of the form

$$a + bi,$$

where a and b are real numbers and i is a (nonreal) number having the property that

$$i^2 = -1.$$

If $b \neq 0$, the complex number $a + bi$ is called an *imaginary number.* The quadratic equation

$$x^2 - 2x + 5 = 0,$$

for example, has as its solutions the imaginary numbers

$$1 + 2i \quad \text{and} \quad 1 - 2i.$$

Since we are primarily concerned with real numbers in this book, we shall usually drop the word "real" and just write "number" for "real number."

Four operations are defined on the system of real (or complex) numbers, namely addition, subtraction, multiplication, and division by a nonzero number. The basic properties of these operations are well known to the readers of this book. They include, for example, the following: for any numbers a, b, and c,

$$a + b = b + a, \qquad ab = ba,$$

$$a + (b + c) = (a + b) + c, \quad a(bc) = (ab)c,$$

$$a(b + c) = ab + ac,$$

$$a + 0 = a, \qquad a \cdot 1 = a,$$

$$a + (-a) = 0, \quad a\left(\frac{1}{a}\right) = 1 \qquad \text{if } a \neq 0.$$

If a and b are numbers such that $ab = 0$, then either $a = 0$ or $b = 0$. Conversely if either $a = 0$ or $b = 0$, then $ab = 0$. These two statements may be combined into one as follows:

$$ab = 0 \quad \text{if and only if} \quad a = 0 \text{ or } b = 0.$$

2. *Inequalities*

An important property of the system of real numbers is that the set (or collection) of nonzero numbers can be separated into two parts, one part made up of the positive numbers and the other part the negative numbers. Thus each real number is either a positive number, zero, or a negative number.

Two nonzero numbers a and b either *agree in sign* (that is, both are positive or both are negative) or *differ in sign* (that is, one is positive and one is negative). If a and b agree in sign, ab and a/b are positive numbers, whereas if a and b differ in sign, ab and a/b are negative numbers. The nonzero numbers a and $1/a$ always agree in sign; a and $-a$ always differ in sign. The sum of two positive numbers is positive, of two negative numbers is negative.

If $a \neq b$, one of the numbers a and b is greater than the other. We write

$$a > b$$

if a is greater than b, and

$$a < b$$

if a is less than b (which is to say that b is greater than a). The symbols $<$ and $>$ may be defined as follows:

1.1 Definition. If a and b are real numbers, then

(i) $a > b$ if $a - b$ is a positive number.
(ii) $a < b$ if $a - b$ is a negative number.

An expression of the form

$$a > b \quad \text{or} \quad a < b$$

is called an *inequality*. For example,

$$9 > 5 \quad \text{and} \quad \frac{13}{2} < 7,$$

since $9 - 5 = 4$, a positive number and $\frac{13}{2} - 7 = -\frac{1}{2}$, a negative number. Clearly

$$a > b \quad \text{and} \quad b < a$$

have the same meaning.

If a is a positive number, $a - 0$ is positive and $a > 0$ according to the definition above. Conversely, if $a > 0$, then $a = a - 0$ is a positive number. Therefore,

the number a is positive if and only if $a > 0$, and, similarly,

the number a is negative if and only if $a < 0$.

The following laws of inequalities will be useful in the sequel.

1.2 If $a > b$ and $b > c$, then $a > c$.

1.3 If $a > b$, then $a + c > b + c$ for every number c.

1.4 If $a > b$ and c is positive, then $ac > bc$.

1.5 If $a > b$ and c is negative, then $ac < bc$.

Particular note is to be made of the distinction between 1.4 and 1.5. Thus multiplication by a positive number maintains the direction of the inequality, whereas multiplication by a negative number reverses the direction of the inequality.

These laws may be proved by using the properties of positive and negative numbers and Definition 1.1. We illustrate this fact by proving one of the laws.

Proof of 1.4: Since $a > b$, both $a - b$ and c are positive numbers. Hence $(a - b)c$, which is equal to $ac - bc$, is a positive number, and $ac > bc$ according to 1.1.

The laws 1.2–1.5 remain valid if the direction of each inequality is reversed. We restate these laws after this reversal:

1.2′ If $a < b$ and $b < c$, then $a < c$.

1.3′ If $a < b$, then $a + c < b + c$ for every number c.

1.4′ If $a < b$ and c is positive, then $ac < bc$.

1.5 If $a < b$ and c is negative, then $ac > bc$.

We may write

$$a < x < b \quad \text{or} \quad b > x > a$$

if $a < x$ and $x < b$. In this case, x is a number between a and b. In a continued inequality such as this, the inequality signs will always have the same direction. Thus, we shall never write

$$a < x > b \quad \text{or} \quad a > x < b.$$

Other useful symbols are $\geq$ and $\leq$ which are defined in an obvious way:

$$a \geq b \quad \text{if either} \quad a > b \quad \text{or} \quad a = b,$$

and similarly for $a \leq b$. A continued inequality such as

$$a \leq x < b$$

indicates that either x is between a and b or $x = a$.

Example 1. Solve the inequality $7x - 5 > 3x + 4$.

Solution: If x is a number such that

$$7x - 5 > 3x + 4,$$