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Summary

When a linear elliptic partial differential equation has singularities in the boundary conditions, it is generally impossible to obtain accurate approximations in the neighbourhood of those points using standard numerical methods . In practical engineering problems, quantities involving derivatives such as stresses, bending moments, shear forces, etc.... play an important role in the design process, so that an accurate knowledge of the behaviour of these quantities, especially near the singular points, is necessary .

In this text, a method is outlined to obtain an accurate approximation in the domain and on the boundary, especially near the singularities . In particular, any derived quantity of the proposed approximate solution can be obtained, with sufficient precision, by analytical derivation .

The method is based on a set of singular functions satisfying the partial differential equation and part of the boundary conditions . The method is developed in detail in case of potential problems with mixed boundary conditions, and for thin plate problems, with any combination of simply-supported, clamped or free boundary conditions . Special attention is given to skew plates .

Principal notations

N. B. : References to the bibliography are indicated between brackets using the first two characters of the author's name followed by a number .

x, y, z	Rectangular Cartesian coordinates
r, θ	Polar coordinates
$\bar{x}$	Spacial coordinates of a point in the domain or on the boundary
α	Angle of skew
β	Angle of twist
ν	Poisson's ratio
q	Normal surface force, per unit of area
D	Flexural rigidity
G	Modulus of rigidity
τ_{xz}, τ_{yz}	Shearing stress components in the rectangular coordinate system
u, v, w	Components of displacement in the rectangular coordinate system
M_x, M_y, M_{xy}	Bending and twisting moments with respect to the rectangular coordinate system

M_n, M_s, M_{ns}	Bending and twisting moments with respect to the normal and tangential directions to the boundary
Q_x, Q_y	Normal shears with respect to the rectangular coordinate system
Q_n	Normal shear with respect to the normal direction to the boundary
Δ	Laplace operator
$\Delta\Delta$	Biharmonic operator
n	Outward normal to the boundary
$\mathcal{L}$	Linear partial differential operator
B	Boundary conditions differential operator
S	Essential boundary conditions differential operator
G	Natural boundary conditions differential operator
Ω	Domain
Γ	Boundary
Γ_s	Essential boundary
Γ_g	Natural boundary
Γ_R	Remaining boundaries
Γ_A	Boundaries adjacent to a singular point
Γ_D	Dirichlet boundary

Γ_N	Neumann boundary
R	Residual error
R_L	Residual error of the differential operator
R_s	Residual error in the essential boundary conditions
R_g	Residual error in the natural boundary conditions
a_i^k, a^k, A_i^k, A^k, A_λ, B_λ, ...	Unknown parameters
w, W	Weighting functions
λ	Real or complex eigenvalues
Q	Orthogonal matrix

Introduction

The problems of mathematical physics, such as electrostatics, quantum mechanics, elasticity theory, hydrodynamics etc.... usually lead to partial differential equations, more rarely to ordinary differential equations . These equations have to be integrated under the initial or boundary conditions of a specific problem . The necessity of solving these problems as accurately as possible, led, in the past two decades, to the development of very powerful numerical solution techniques such as the finite difference method, the finite element method and the more recent boundary (integral) element method .

These methods came as a natural consequence of the development of the computer, which was able to handle problems involving large amounts of numerical storage and manipulations . Some of the basic principles of these methods (KA1,CO11) can be traced to pre-computer times and involve different ways of solving the governing equations of a problem, i.e. Galerkin, collocation, least squares techniques, etc .

One of the first numerical methods ever used was the finite difference method . The partial differential equation under consideration was approximated by equations in finite differences, obtained from the differential equation by replacing the derivatives and the differential operators by approximate expressions in terms of difference ratios or function values in separate points of a net . An interpolating polynomial is constructed, which takes the same values in the netpoints as the unknown function . Then the derivatives of the unknown function are considered to be approximately equal to the corresponding derivatives of the interpolating polynomial .

In the standard Rayleigh-Ritz method the approximate solution is expected to satisfy a priori exactly all the 'essential' (rigid or geometric) boundary conditions on that part of the boundary where such conditions are specified . This leads to an 'unadapted' choice of functions describing the field problem . This difficulty increases when irregular forms of the domain are considered . The finite element method (e.g. ZI2) diminishes these disadvantages of the Rayleigh-Ritz method by dividing the continuum into a series of elements of 'elementary' form . Consequently the choice of the approximate functions and the adaption to complicated domains becomes easier .

On the other hand, Trefftz (TR1) considered an approximate solution based on parametric functions satisfying a priori the differential equation of the problem . On this principle are based the so-called boundary solution methods, whose common feature is that unknown parameters are evaluated in such a manner that the proposed solution satisfies the boundary conditions 'as well as possible' (ZI1, BR1, KE4) .

These boundary solution procedures have specific merits . In particular, they may easily be applied to semi-infinite regions, and to singularities, such as those due to isolated loads, can be properly taken into account . Moreover, it is now generally recognized that for simple situations the boundary solution procedures are more economical than the standard finite difference or finite element solutions . Therefore, various coupling schemes have been suggested (ZI1) to combine the boundary solution procedure with the conventional finite element process . Together with the coupling of the finite element method and the boundary solution procedures new types of finite elements have appeared .

Tolley (TO1) introduced the Large Singular Finite Element Method, based on a set of parametric functions in the angular points of the region . These functions satisfy the governing differential equations in the domain and the external boundary conditions adjacent to the angular point under consideration . The domain is divided into as many elements as (angular) singularities occur . Unknown parameters of the functions are determined by imposing interelement continuity relations .

Jirousek (JI1, TE2) introduced a second type of large finite elements by replacing the Ritz-method by the Trefftz-method in the treatment of the elements of the domain . These type of elements gather the specific advantages of the Trefftz-method (e.g. treatment of singularities) and the facilities of existing finite element procedures .

Already in the 1960's Conway (CO1-CO9) published several articles using a method of approximating the boundary conditions by a truncated set of functions satisfying the governing equations in the continiuum .

Conway gave this method the name "point-matching" or "modified point-matching" method where the boundary conditions are met in some average sense, such as least squares . "Boundary collocation" method is also a name which is used .

In this text, some specific elliptic partial differential equations of order 2 or 4, with singularities in the boundary conditions are studied . The goal is to develop a solution method, capable of finding approximate solutions as well as the approximate partial derivatives of a given problem .

Applying such a method to e.g. elasticity problems, such as the study of thin plates, allows us to approximate stresses, bending and twisting moments, shearing forces etc. . These quantities are in general difficult to determine accurately with standard numerical methods .

The fact that singularities occur in the boundary conditions, implies that the method must be able to approximate 'correctly' the exact solution and its derivatives in and near the singularities themselves .

We confined ourselves to two types of elliptic problems, namely the Laplace and the biharmonic problem . These problems are treated in polygonal domains with different combinations of the boundary conditions . It is important to note that special attention was given to problems with non-symmetrical geometry and boundary conditions, containing at least one singular point .

These two specific equations span a large class of engineering problems . We cite, e.g. potential problems, torsion of a rigid beam, membranes, temperature distribution in a continuum, incompressible fluid flow in channels, Airy-stress functions, bending of thin plates, etc....

The problems treated hereafter describe stationary phenomena in which the quantities of interest do not vary with time or, at least, are of negligible variation . But we are convinced that the method outlined in this work can be extended to problems in which time variation is appreciable .

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Chapter One
Singularities

Reasonable approximations to the solutions of the boundary value problems of mathematical physics can usually be found by standard numerical methods, as long as the problems contain no singularities. However, when some form of singularity is present, as is often the case with mixed boundary value problems, the classical numerical methods are generally found to be unsatisfactory. Thus if such methods are to be used, some special treatments are necessary in the neighbourhood of the singularity. Consequently, considerable attention has been given to seek modifications of such methods and to derive new methods or new elements which give special treatment to singular points.

An alternative is to use some specific form of analytical methods. Such methods are not as flexible as the numerical methods, but they can sometimes be very efficient in special cases.

1.1 Classification of singularities.

1.1.1. Interior singularities.

Interior singularities (AM1) occur when one, or more than one, coefficient of the partial differential equation becomes singular at points P_i inside the integration domain. In such problems the solution will usually have a singularity at P_i.

Physical problems where these singularities occur are source-sink and concentrated point-load problems.

For example, singularities inside the region occur in a Poisson equation

$$\Delta u = f(x,y) \tag{1.1.}$$

if the function f is "not well behaved" . A typical source problem involves the case

$$f(x,y) = 0 \tag{1.2.}$$

except at the origin whereupon

$$u = A \log r \tag{1.3.}$$

with $r^2 = x^2 + y^2$ (1.4.)

Another example is the deflection of a thin plate by a concentrated point-load at the origin . The deflection u is solution of the biharmonic equation

$$\Delta\Delta u = f(x,y) \tag{1.5.}$$

where

$$f(x,y) = 0 \tag{1.6.}$$

except at the origin .

A solution u of such problems generally contains expressions such as

$$r^2 \log r \tag{1.7.}$$

for which the second derivatives do not exist at r = 0 .

1.1.2. Boundary singularities .

Boundary singularities occur when the function imposed on the contour has singular points . One of the most 'violent' of these is a finite difference (jump) in function value in some isolated points . But weaker types of singularities can occur whenever certain lower derivatives are discontinuous in some points .

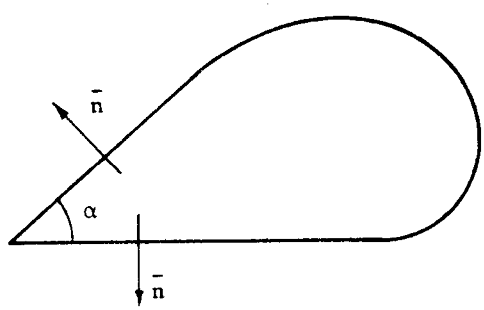

Fig. 1.1. : Change of direction of the boundary .

A sudden change of direction of the boundary (Fig. 1.1.) generally introduces a boundary singularity . These singularities occur whenever polygonal regions are considered .

In mixed boundary-value problems these types of boundary singularities occur usually in the angular points of the region, where, together with a sudden change of direction of the boundary, a jump in function value or a change in boundary conditions can occur .

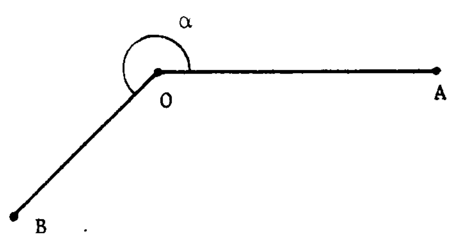

Fig. 1.2. : Typical corner .