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Preface

This book is written for three groups of readers:

(a) Students of physics, engineering and other applied sciences who have completed a conventional course in differential equations. Many of these students may realise that the subject is of great practical importance but have no confidence that they will be able to master the formal mathematics required.

(b) Professional scientists and engineers who were once in the former category, but now realise that mathematical models of real systems do throw up differential equations; and that such equations have a nasty habit of being different from the textbook examples. After an hour or so grappling with his old notes and a pile of authoritative but bewildering treatises, the unhappy member of this group goes in search of a mathematician or uses a computer and standard software that he only partly understands. In either case he has isolated himself from the process of solving the equation, and as a result has diminished his understanding of what the formal or computed solutions mean.

(c) Teachers who need to explain the physical significance of equations and their solutions in plain language to students of applied science and engineering whose interests and skills are more often in experimental work than in mathematics.

The common problem for people in these three groups is that formal mathematics is an obstacle rather than an aid to understanding.

While it has long been recognised that the ability to make order-of-magnitude numerical estimates is an essential skill of the working scientist and engineer, there is no corresponding method for a 'back-of-envelope' treatment of differential equations. What is needed is a method that bypasses the mathematical difficulties and also emphasises those features of the solution most useful in the design of practical devices and systems. It is the prime purpose of this book to fill this need.

The reader will be relieved to find that this does not require the use of any formal mathematics beyond that learned in sixth-form or first-year university courses. Instead, much greater use is made of physical and intuitive arguments of a kind familiar to experimental scientists. We shall often appeal to analogy, prior experience and experimental evidence, and make use of inference and hypothesis. Using this approach, the reader who cannot afford to carry a great weight of specialised mathematical equipment will find he can understand, and therefore enjoy, mathematical models whose equations are too difficult to solve by other means.

Contents

* Sections marked with an asterisk may be omitted at a first reading.

Chapter 1

Introduction

1.1 Approximation with understanding as its goal

Mathematical models usually describe either processes or devices. In processes, the system is changing with time; in devices, the changes studied may also be with time, or the interest may be in the way in which variables such as temperature or concentration change with position within the device. In both cases, rates of change are involved, so the mathematical model leads to a differential equation. The principal object of this book is to solve differential equations so as to obtain a broad understanding of the systems they describe, using a minimum of formal mathematics.

Differential equations may be classified in order of increasing difficulty:

(a) Very simple equations with simple and well understood solutions.

(b) Equations that can be solved exactly, but whose solutions are so complicated, or contain such unfamiliar functions, that only a specialist can understand their physical significance.

(c) Equations that cannot be solved at all by formal mathematics. There are no useful exact solution methods, for instance, for the majority of nonlinear equations.

The equations used for teaching and examination purposes are deliberately chosen mainly from the first class, with occasional examples from the second. Unfortunately, many equations arising from practical problems belong to the second or third classes, and their formal solution is either impossible or physically obscure.

The key to physical understanding of these equations is the use of approximation. Exact solutions are complicated and may need unfamiliar functions because, by definition, they must describe exactly every detail of the changes in the variables. By contrast, approximate solutions can strip away the overlying detail to show the essential relationships between the physical variables. What is more, these relationships can be expressed in symbols and words that are familiar to all scientists and engineers.

As an illustration, consider an equation that can be used as a mathematical model for the chemical reaction:

$$A + 2B \rightarrow C.$$

The equation is

$$\frac{dn}{dt} = K(a-n)(b-2n)^2 \tag{1.1}$$

where a and b are the numbers of molecules A and B at $t=0$, and n is the number of molecules C after time t.

The exact solution is

$$t = \frac{1}{K}\left[\frac{1}{2a-b}\left(\frac{1}{b-2n} - \frac{1}{b}\right) + \frac{2}{(2a-b)^2}\ln\left(\frac{1-2n/b}{1-n/a}\right)\right]. \tag{1.2}$$

This result is too complicated to give a direct answer to the most practical questions about the reaction, such as 'What does the curve of n against t look like?' and 'How does the shape of this curve depend upon a, b and K, the constants in the equation?' By contrast, the approximate solution† (assuming the reaction ends when all the molecules B are used up) answers these questions directly. It is

$$n \approx \tfrac{1}{2}b(1 - e^{-t/\tau}) \tag{1.3}$$

where

$$\tau = \frac{1}{Kab(1 - b/4a)}. \tag{1.4}$$

In words, solution (1.3) states that 'n increases approximately exponentially from zero at $t=0$ to a final value $n = \frac{1}{2}b$, with a time constant given by formula (1.4)'.

The exponential curve and its time constant are familiar ideas in science and engineering, and the curve of n against t (figure 1.1) can be pictured and sketched immediately, using either the mathematical description (1.3) or its verbal equivalent.

The answer to the second question, about the effect of changing a, b and K, is given in the compact formula (1.4) for the time constant τ. In words, 'τ is inversely proportional to a, b and K, subject to a correction that has a relatively slow variation with the ratio b/a'. Formulae like (1.4), which show the way in which the most important parameters of the system depend upon each other, will be called *design formulae*, since they embody the essential relationships needed by engineers who have to design practical systems.

The example shows what is meant by solving for physical understanding. What is required first is a solution that can be pictured graphically and described in simple and familiar words; the second requirement is a design formula showing the most important relationship between the parameters. In

† Derived in problem 3.6.

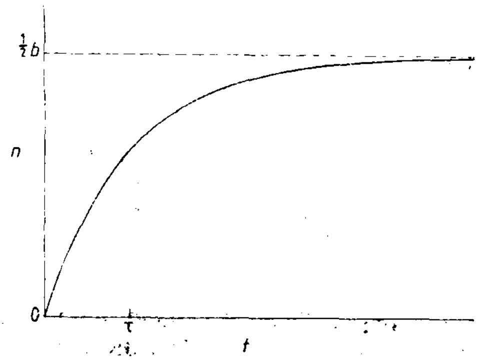

Figure 1.1 Approximate solution of equation (1.1).

the conventional methods of solution, a mathematical solution is obtained first, and from this graphs can be plotted and design formulae derived. This means that the graphs and the design formulae, which may be quite simple to understand, are obtained as secondary products of the primary mathematical investigation, which may be very complicated.

The special feature of the approximate method of solution used in this book is that it reverses this procedure. The first step is to use physical rather than mathematical arguments to make a qualitative sketch (QS for short) of the likely form of the graphical solution; the second step is to use the method of trial function (or TF) approximation to add more detail to the graph, and also to find the design formula directly. This qualitative sketch/trial function (QSTF) method thus goes straight to what is needed for physical understanding, instead of using the conventional roundabout route. The consequence is that the method is generally quicker and involves fewer mathematical steps than exact procedures. A further consequence is that, by avoiding the formal mathematical difficulties, this approximate approach works for many equations that are impossible to solve exactly in closed form.

1.2 Trial function approximation

Since trial function approximation, which is the second step of the QSTF method, may be unfamiliar, we shall first describe it in brief outline, leaving all the details of the steps to be filled in later. It is started by choosing a mathematical function, called the *trial function* (or TF for short), which is believed to be a good approximation to the exact solution; this TF always contains one or more *unknown parameters*. The second step is to combine the TF and the equation, using a process called *residual minimisation*, to find formulae for the unknown parameters in terms of the physical constants of the system being studied.

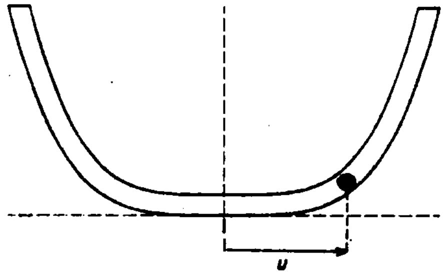

Figure 1.2 A ball-bearing oscillating in a smooth tube bent to produce a restoring force proportional to the cube of the displacement.

As an illustration, consider the motion of a ball-bearing oscillating in a glass tube that is bent into a curve such that the restoring force depends upon the cube of the displacement u (figure 1.2). The governing equation, ignoring frictional losses, is

$$\frac{d^2u}{dt^2}+cu^3=0 \tag{1.5}$$

and the auxiliary conditions are that the ball-bearing is released from rest at a displacement u_0 when $t=0$. Expressed mathematically, this is

$$u(0)=u_0 \qquad u'(0)=0. \tag{1.6}$$
[initial velocity zero]

The goal of solving this mathematical model is to find how the frequency f of the oscillation depends upon the physical parameters; in other words, the goal is a design formula for f. The methods for choosing trial functions will be explained later. In this case the TF used is

$$u^*=u_0\cos\omega t. \tag{1.7}$$

This can be seen to fit the auxiliary conditions (1.6) and to be a physically reasonable way of describing an oscillation of amplitude u_0. The unknown parameter is the angular velocity ω.

The second stage is to find an approximation for this unknown parameter by *residual minimisation*. Again the details of this process will be discussed in great detail later; for the present, the important point is that it involves only a few lines of simple mathematics to reach the result

$$\omega=0.87c^{1/2}u_0. \tag{1.8}$$

The relationship between frequency and angular velocity is

$$f=\omega/2\pi \tag{1.9}$$

so the design formula for f is

$$f=\frac{0.87}{2\pi}c^{1/2}u_0=0.14c^{1/2}u_0. \tag{1.10}$$

This is the required design formula relating the most important characteristic of the performance of the oscillator, i.e. frequency f, to the physical constants c and u_0 present in the mathematical model.

The directness and simplicity of TF approximation contrasts with the tediousness of the formal method of solution. First, the exact solution of the equation, involving elliptic functions, would have to be found; then the rather unusual periodic properties of this solution recognised; and finally, not without some difficulty, the period found from appropriate tables. After all this, substantially the same design formula is obtained, differing only in that the numerical constant would be 0.135 instead of 0.14.

1.3 The qualitative sketch

The brief account of the TF approximation in the last section will have raised many questions. The first step, for instance, is to choose a TF that is believed to be a good approximation to the exact solution, and it is by no means clear how this is to be done. The rather scattered descriptions of the TF method in the literature are of little help; they do not describe any general technique for choosing a TF, and physical understanding is seldom their prime objective. We shall show in this and the next section that, for physical equations that describe real processes or devices, there is a systematic method for finding a suitable trial function.

The key to the problem is that for physical equations it is almost always possible to sketch a graph of the solution without using the equation at all. In the case of the chemical reaction discussed in §1.1, for instance, it is clear without any mathematics that the number n of molecules C must start from zero and increase with time. Intuition also suggests that the curve of n against t must be smooth, since there are no mechanisms to cause any humps or sudden changes in direction. Finally it is clear that the number of molecules C is eventually limited by exhaustion of one of the reactant molecules A or B. Putting these three statements together gives a picture of a smooth curve rising from $n=0$ and eventually flattening off at a steady value. When such a curve is sketched its shape must look like figure 1.1. Without any reference to the equation, therefore, we have been able to sketch the approximate shape of the solution curve, using only the initial conditions and physical commonsense.

A sketch of the expected solution made in this way without formally solving the equation will be called a *qualitative sketch*. It is drawn by first carefully considering any auxiliary conditions, which will usually fix the beginning and end of the curve, and then using physical intuition to fill in the intervening curve. In Chapter 2 and in later examples we shall show how this physical intuition can be supplemented by analogy with known solutions of simpler equations and direct inspection (without formal solution) of the equation itself.

The drawing of the qualitative sketch is the most important single step in solving with physical understanding. It serves three purposes:

(a) It goes a long way to answering the question. 'What does the solution look like?'

(b) It forces the solver to think in the first instance about the physics of the real apparatus, rather than about the abstract mathematical equation.

(c) As we shall show in the next section, it provides a systematic way of choosing the TF.

Two more ingredients are needed to complete our physical understanding, as defined in §1.1. First, although the qualitative sketch gives the shape of the curve, the scale of one or other of the axes is usually missing. Secondly, the appropriate design formula for the problem, relating the performance to the constants of the system, is wanted. Both these needs are supplied by the TF approximation, and we shall next show how the qualitative sketch makes the choice of TF more systematic. The two stages, the making of the qualitative sketch (or QS), followed by the TF approximation, together make up the QSTF method of solving with physical understanding.

1.4 Standard functions

When the qualitative sketches are made for a number of practical problems chosen from various fields, it turns out that they almost always look like one of the curves in figure 1.3(*a*)–(*e*). These curves may be classed as either sinusoidal (figure 1.3(*a*)), exponential (figure 1.3(*b*) and (*c*)) or parabolic (figure 1.3(*d*) and (*e*)). For some less common auxiliary conditions, the curves are displaced. An example is shown in figure 1.3(*f*), which is just the curve of figure 1.3(*b*) displaced upwards and to the right. The three shapes are so distinctive that they can be immediately recognised even when they are displaced, and the curve of figure 1.3(*f*) can be classed as exponential without any difficulty. There are good physical reasons why these graphical shapes are so common as solutions to physical equations. The sinusoid, with its repeated maxima and minima, is the simplest curve that can represent an oscillation of constant amplitude and frequency, and such oscillations occur in every branch of science. Similarly, the exponential, characterised by its long plateau, is the simplest curve that can describe a smooth change ending by becoming asymptotic to a steady state, behaviour typical of changes of state occurring in many different sciences. Finally, the parabola is the simplest curve characterised by symmetry about a single extremum, and such symmetry is a common factor in a whole variety of apparatus and devices. This classification of

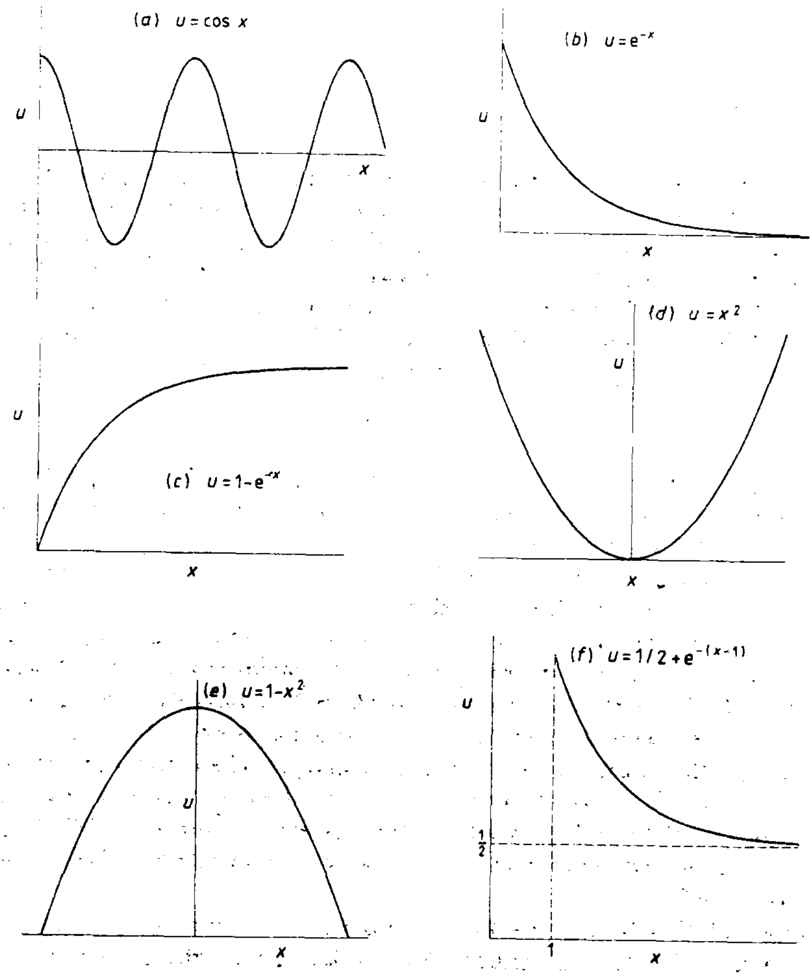

Figure 1.3 Curves of familiar functions that commonly occur as the solutions to physical equations: (*a*) sinusoidal, (*b*) and (*c*) exponential, (*d*) and (*e*) parabolic, and (*f*) displaced exponential.

qualitative sketches into one of three classes gives a systematic basis for the choice of TF.

The functions used to generate the curves in figure 1.3 cannot be used directly as TFs since they do not contain any unknown parameters, which are essential for the final step of the TF approximation. The simplest useful forms of TF corresponding to the three kinds of curve are:

sinusoidal

$$u = A \cos \omega t \tag{1.11}$$

exponential

$$u=A\,e^{-t/\tau} \qquad u=A(1-e^{-t/\tau}) \tag{1.12}$$

parabolic

$$u=A(1-x^2/l^2). \tag{1.13}$$

These each contain two parameters that have a clear physical meaning: A is the amplitude of the cosine wave and for the exponential and the parabola it is the height of the curve; ω, τ and l respectively correspond to the frequency ($f=\omega/2\pi$) of an oscillation, the time constant of a change and the half-width of a symmetrical curve. If the qualitative sketch is displaced, then constants must be added to these simple trial functions to shift the origin of the axes. The displaced curve of figure 1.3(f), for instance, corresponds to

$$u=u_\infty+A\,e^{-(t-t_0)/\tau}$$

where $u_\infty=\frac{1}{2}$ and $t_0=1$ are constants describing the shift of origin. When examples are studied, it will be seen that such additional constants always have a clear physical interpretation.

The cosine, exponential and parabola, in either their simple or displaced forms, will be called the three standard functions of approximation or more simply the *standard functions*. They form the basis both for studying the QSTF method and for using it to solve equations. The technique of the QSTF method is best learnt by first mastering problems for which the standard functions can be used as TFs. This is the method followed in this book, in which the technique is first given with worked examples for exponential TFs (Chapters 3 and 4), and then for parabolic (Chapter 5) and cosine TFs (Chapters 6 and 7). Once the techniques for the standard functions are mastered for ordinary differential equations, it is possible to extend their use to partial differential equations (Chapters 8–10).

At the end of each chapter a number of exercises (for practising techniques) and problems (for applying them) are given. Like the examples in the main text, these are carefully chosen to avoid redundancy, and they often introduce new points or extensions of the method. Some of the working in the later chapters presupposes that examples in earlier chapters have been worked through; in particular, it is impossible to understand the chapters on partial differential equations without a firm grasp of the QSTF method applied to ordinary differential equations.

In the examples, the qualitative sketch is first drawn and recognised as having one of the three standard shapes; it is then compared with graphs of the corresponding standard functions in their simple and displaced forms. The graph that matches the qualitative sketch is picked out and the corresponding function is the TF for the problem. This procedure automatically leads to useful results. The qualitative sketch ensures that the corresponding TF is physically reasonable and conforms to the auxiliary conditions. The use of the standard functions means that any mathematics will be relatively straightforward and familiar, and therefore clearly understood. Finally, because the standard functions contain only the most important parameters, their use guarantees that the TF approximation will yield a design formula of direct practical importance.

As mentioned in §1.2, the last stage of the TF approximation is residual minimisation. This stage can be made quite simple, but the technique is best learnt by following some examples, and is therefore first introduced in Chapter 3. The techniques for the three standard functions differ only in small details, so that the method becomes more and more familiar as more equations are studied. This is in direct contrast to the methods of formal mathematics, in which new and often highly specialised techniques and functions have to be learnt as each new type of physical equation is encountered.

Once the QSTF method using standard functions is mastered, it is possible to tackle some of the less common problems that need nonstandard TFS (see Chapter 11). Very often, however, the standard TFS can be adapted slightly to meet the new circumstances, and a number of examples of such adaptations are given in the appropriate chapters alongside standard problems.

1.5 Useful accuracy

It is seldom in applied science or engineering that exact solutions are required. This is for two reasons. The first is that the equation itself is not an exact description of the system that it models. Some terms may be omitted because they are 'small', which may mean anything from 10^{-6} to one-third of the total. Other terms are approximate because they include physical parameters or functions that are themselves approximate. For example, in an equation of heat power balance, loss by radiation and conduction might be neglected, while the convective loss is described by a term based on an empirical law of dubious validity, involving a parameter—the heat transfer coefficient—which is not known accurately and which depends on several uncontrolled factors.

The second reason why exact solutions are seldom required by the applied scientist or engineer is that the use to which the solutions are to be put does not warrant it. One of the arts of good applied science is to achieve accuracy *adequate for a particular purpose.* The pursuit of excessive accuracy is both time-consuming and costly. It is a strange fact that this is recognised in experimental science, where one more significant figure may increase the cost of a measurement 10-fold, whereas, in calculations performed on the same system, much time is often wasted in seeking accurate solutions to inaccurate governing equations. It is as if the mathematical form in which the problem is expressed exerted some kind of hypnotic influence over the solver, tempting him to pursue solutions that are mathematically respectable but physically unrealistic or even meaningless.

In the early stages of experimental work, particularly in the course of