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ABSTRACT

By using pullbacks, we obtain a description of finitely generated modules over the integral group ring of a non-abelian group of order pq. The description is detailed enough to obtain information about the behavior of the modules in direct sums. We make the description more precise by relating it to the locally free class group of the integral group ring.

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INTRODUCTION

Let G be a non-abelian group of order pq (p and q distinct primes). In this paper we solve the problem of describing all isomorphism classes of finitely generated left ZG-modules and the behavior of these modules in direct sums, where ZG is the integral group ring of the group G. We consider ZG-modules in general and not merely ZG-lattices; that is, we do not assume that modules are torsion-free as additive groups.

The motivation for this work is that there seems to be no non-commutative, non-hereditary, non-artinian noetherian ring all of whose finitely generated modules are known. The technique used here is to represent the ring ZG as a pullback (in chapter 1), and then adapt the methods of Levy [2], [3], and [4].

Our main result (in chapter 10) is to define a function "class" cl( ) from the category of (finitely generated left) ZG-modules into the (finite) set CLS(ZG) of isomorphism classes of fractional (not necessarily full) left ZG-ideals in the quotient ring QG of ZG. The function cl( ) is defined in such a way that, for arbitrary finitely generated left ZG-modules M and N,

$$(1)\quad M \cong N \text{ if and only if } \begin{cases} M_t \cong N_t \text{ (localization at t) at all primes t, and} \\ \mathrm{cl}(M) = \mathrm{cl}(N). \end{cases}$$

(This is analogous to the situation for modules over Dedekind domains, for which a finitely generated module is determined up to isomorphism by its localization at all prime ideals and its Steinitz class.) In addition, cl( ) is defined so that, if H is a fractional ZG-ideal in the class cl(M) for some finitely generated left ZG-module M, then $H \in \mathrm{cl}(H) = \mathrm{cl}(M)$.

We also define an operation "+" on CLS(ZG) in such a way that

$$(2)\qquad \mathrm{cl}(M \oplus N) = \mathrm{cl}(M) + \mathrm{cl}(N)$$

for any finitely generated left ZG-modules M and N. CLS(ZG) forms a semi-group under this operation and decomposes as the disjoint union of a finite collection of subgroups, where each subgroup itself is just a genus of frac-

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tional ideals. (Two modules are in the same genus if they are locally isomorphic at all primes.)

We show that every non-zero fractional left ZG-ideal is a projective module for some Z-order between ZG and its quotient ring QG. Those isomorphism classes of fractional ideals which are in the image of cl( ) are actually locally quasi-free. (See corollary 10.20 for definitions.) Hence each genus in the image of cl( ) forms a locally quasi-free class group, and we show that its order divides the order of the locally free class group of ZG.

In chapter 11 we apply this numerical information, along with (1), to questions concerning local versus global isomorphism and direct sum decompositions of finitely generated left ZG-modules. We show that the Krull-Schmidt theorem fails for direct sum decompositions of finitely generated ZG-modules (for G as above), but it usually holds for direct sum decompositions of finitely generated modules over localizations Z_tG. (See theorems 11.1 and 11.3.)

We also use (1) to show that the following are equivalent for finitely generated ZG-modules (see theorem 11.4):

(a) Modules cancel from direct sums of modules.

(b) Lattices cancel from direct sums of lattices.

(c) Free modules cancel from direct sums of lattices.

(d) $q = 2$.

Even though ordinary cancellation fails for $q \neq 2$, we know, by Goodearl [1], that power cancellation holds for finitely generated ZG-modules. (See also Guralnick [2].) That is, if $M \oplus X \cong M \oplus Y$ for finitely generated ZG-modules M, X, and Y, then $X^{(d)} \cong Y^{(d)}$ for some d. In theorem 11.8, we show that a single exponent d works for any choice of M, X, and Y, where $d = 1$ if $q = 2$, and $d = q$ if $q > 2$. In corollaries 11.10 and 11.11 we show that the cardinality of the genus of each finitely generated ZG-module M divides the order of the locally free class group of ZG, and if M and N are in the same genus, then $M^{(e)} \cong N^{(e)}$, where e divides the order of the locally free class group of ZG. (See also Guralnick [1], [2].) This gives a second proof that power cancellation holds, but the bound e obtained is not as sharp as the bound d obtained above.

As a final application, we compute the projective dimensions of finitely generated ZG-modules without artinian direct summands. For such a ZG-module M, we get that $\mathrm{pdim}(M) \leq 1$ if $\mathrm{cl}(M)$ is projective, and $\mathrm{pdim}(M) = \infty$ otherwise. (See theorem 11.15 and corollary 11.16.)

The proofs of (1) and (2) above are given in chapter 10 and rely on the structure theory developed in chapters 1 through 9, which we now outline.

In chapter 1 we express ZG as a certain subring of the hereditary ring $Z \oplus Z[\zeta_q] \oplus \Lambda$, where ζ_q is a primitive qth root of unity, and Λ is a certain (noncommutative) hereditary noetherian prime ring.

In chapter 2 we introduce the notion of a ZG-diagram, a certain configuration of modules over the coordinate rings Z, $Z[\zeta_q]$, and Λ, together with certain module homomorphisms. The collection of ZG-diagrams forms a category. We define a functor M( ) from the category of finitely generated ZG-diagrams to the category of finitely generated ZG-modules and show that M( ) is a representation equivalence. Thus we reduce the problem of classifying ZG-modules to that of ZG-diagrams.

In chapter 3 we describe finitely generated modules over the coordinate rings Z, $Z[\zeta_q]$, and Λ, summarizing results of Rosen, Eisenbud, and Griffith.

In chapter 4 we convert the problem of classifying ZG-diagrams to a matrix problem by fixing direct sum decompositions of the modules that occur in these diagrams and replacing homomorphisms by matrices over fields. We prove a theorem which describes the set of matrix operations which can be performed on the matrices of a diagram without changing the isomorphism class of the diagram. Thus the matrix problem is to describe some canonical form of the matrices of a diagram using the given matrix operations. In chapter 5 we consider the effect on ZG-diagrams of localization and completion at the primes of Z. In chapter 6 we translate into our notation the solution of the matrix problem in Klingler and Levy [1].

Using these results, in chapter 7 we give an explicit description of all finitely generated indecomposable artinian ZG-modules. Since artinian modules are well-behaved in direct sums, we can, for the most part, ignore them, and so in chapters 8 and 9 we give an explicit (and unfortunately quite technical) description of all finitely generated indecomposable ZG-modules without artinian summands and characterize their behavior in direct sums. To apply these results, in chapter 10 we restate this behavior more conceptually in terms of projective modules of Z-orders between ZG and $Z \oplus Z[\zeta_q] \oplus \Lambda$.

For the remainder of this paper we fix the group G and the primes p and q as in the following proposition.

Proposition 0.1: If G is a non-abelian group of order pq, where $q < p$ are primes, then $q \mid (p - 1)$ and

$$(3) \qquad G = \langle x,y \mid x^p = y^q = 1,\ yxy^{-1} = x^k \rangle.$$

where k is a primitive qth root of unity modulo p. Conversely, if p and q are primes such that $q \mid (p - 1)$, then there exists a unique non-abelian group G of order pq (up to isomorphism) given by (3).

Proof: See Hall [1], theorem 9.4.3.□

Throughout this paper we assume that all modules are left modules, and, except in chapter 2, we assume that all modules are finitely generated.

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CONTENTS

1: ZG AS A MULTIPLE PULLBACK

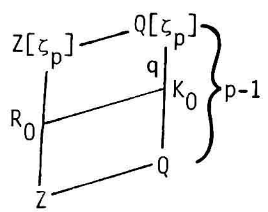
Definition 1.1: Let $R_1, \ldots, R_m$ be rings and let finitely many pairs of ring epimorphisms be given:

$$(1) \qquad R_{i(k)} \xrightarrow{f_k} \!\!\!\!\!\rightarrow \overline{R}_k \leftarrow\!\!\!\!\!\xleftarrow{g_k} R_{j(k)}$$

where $1 \leq k \leq n$ and $1 \leq i(k) < j(k) \leq m$. We call (1) the subdiagram $\mathcal{Q}_k$. We say that the subdiagrams $\mathcal{Q}_1, \ldots, \mathcal{Q}_n$ are independent if, for each c, $1 \leq c \leq m$, the natural map is onto which maps R_c to the direct sum of all those $\overline{R}_k$ to which R_c can be mapped (by the f_k for which $i(k) = c$ and the g_k for which $j(k) = c$). We define the (multiple) pullback of $R_1 \oplus \cdots \oplus R_m$ determined by the independent subdiagrams $\mathcal{Q}_1, \ldots, \mathcal{Q}_n$ to be the ring $R = \{(r_1, \ldots, r_m) \in R_1 \oplus \cdots \oplus R_m \mid f_k(r_{i(k)}) = g_k(r_{j(k)})$ for $1 \leq k \leq n\}$.

In this section we describe ZG as a multiple pullback of three coordinate rings, where two of the coordinate rings are Dedekind domains, and the third is a hereditary order in a simple artinian ring. We begin by describing ZG as an ordinary pullback of a group ring and a skew group ring, where the groups are of prime order q. (Recall that $|G| = pq$ and $q \mid (p-1)$.) For details concerning algebraic number fields and, in particular, cyclotomic extensions, see Curtis and Reiner [2], section 4H.

Notation 1.2: Let ζ_p be a primitive pth root of unity (in the complex numbers). Note that $[Q[\zeta_p]:Q] = p-1$, and $Z[\zeta_p]$ is the ring of algebraic integers in $Q[\zeta_p]$. Recall that we set $G = \langle x,y \mid x^p = y^q = 1, yxy^{-1} = x^k \rangle$, where k is a primitive qth root of unity modulo p. Let $\sigma \in Gal(Q[\zeta_p] \mid Q)$ be such that $\sigma(\zeta_p) = \zeta_p^k$. Let $H = \langle\sigma\rangle \leq Gal(Q[\zeta_p] \mid Q)$, let K_0 be the fixed field of $Q[\zeta_p]$ under H, and let $R_0 = K_0 \cap Z[\zeta_p]$, the ring of algebraic integers in K_0. Since $|H| = q$, we have $[Q[\zeta_p]:K_0] = q$, and we get the following diagram:



We let Λ denote the skew group ring $Z[\zeta_p] \circ H = \sum_{h \varepsilon H} Z[\zeta_p]u_h$ and A denote the skew group ring $Q[\zeta_p] \circ H = \sum_{h \varepsilon H} Q[\zeta_p]u_h$, where multiplication is given by $u_h u_{h'} = u_{hh'}$ and $u_h \alpha = h(\alpha)u_h$ for $h, h' \varepsilon H$ and scalars α in $Z[\zeta_p]$ or $Q[\zeta_p]$. Clearly Λ is an R_0 order in A. A is in fact a crossed-product algebra (see Reiner [1], section 29 for details), and as such A is a simple K_0-algebra. We have $A \cong M_q(K_0)$, the ring of $q \times q$ matrices over K_0, where $Q[\zeta_p]$ is itself a simple left A-module.

<u>Theorem 1.3</u>: The skew group ring $\Lambda = Z[\zeta_p] \circ H$ is a hereditary order in the simple artinian ring $A = Q[\zeta_p] \circ H$.
<u>Proof</u>: See Rosen [1], lemma 1.3.1, or Curtis and Reiner [2], section 34E.□

Now define maps Φ and Ψ in the diagram

(2) $$\begin{array}{ccc} ZG & \twoheadrightarrow & \Lambda \\ \downarrow & & \downarrow \Psi \\ ZH & \xrightarrow{\Phi} & \overline{Z}_p H \end{array} \quad \text{by} \quad \begin{array}{ccc} x & \mapsto & \zeta_p u_1 \\ \downarrow & & \downarrow \Psi \\ 1 & \xrightarrow{\Phi} & 1 \end{array} \quad \text{and} \quad \begin{array}{ccc} y & \mapsto & u_\sigma \\ \downarrow & & \downarrow \Psi \\ \sigma & \xrightarrow{\Phi} & \sigma \end{array}$$

where ZH is an ordinary integral group ring and $\overline{Z}_p H$ a group algebra, $\overline{Z}_p$ denoting the ring of integers modulo p. It is straightforward to check that all of the maps are ring epimorphisms and that the diagram commutes.

<u>Lemma 1.4</u>: Diagram (2) expresses ZG as the pullback of $ZH \oplus \Lambda$ determined by the maps Φ and Ψ; that is, $ZG \cong \{(a,b) \varepsilon ZH \oplus \Lambda \mid \Phi(a) = \Psi(b)\}$.

<u>Proof</u>: See Galovich, Reiner and Ullom [1], section 2, or Curtis and Reiner [2], section 34E.□

The group ring ZH itself can be expressed as an ordinary pullback of Z and $Z[\zeta_q]$, where ζ_q is a primitive qth root of unity.

<u>Lemma 1.5</u>: The diagram

(3) $$\begin{array}{ccc} ZH & \twoheadrightarrow & Z[\zeta_q] \\ \downarrow & & \downarrow g_0 \\ Z & \xrightarrow{f_0} & \overline{Z}_q \end{array} \quad \text{where} \quad \begin{array}{ccc} \sigma & \mapsto & \zeta_q \\ \downarrow & & \downarrow g_0 \\ 1 & \xrightarrow{f_0} & 1 \end{array}$$

expresses ZH as the pullback of $Z \oplus Z[\zeta_q]$ determined by the maps f_0 and g_0, where $\bar{Z}_q$ denotes the ring of integers modulo q. Thus $ZH \cong \{(a,b) \in Z \oplus Z[\zeta_q] \mid f_0(a) = g_0(b)\}$.

Proof: See Levy [3], 1.13.□

Lemmas 1.4 and 1.5 together allow us to view ZG as a subring of $Z \oplus Z[\zeta_q] \oplus \Lambda$ determined by the maps Φ, Ψ, f_0, and g_0 of diagrams (2) and (3). It remains to disentangle the maps to express ZG as a multiple pullback of $Z \oplus Z[\zeta_q] \oplus \Lambda$.

Notation 1.6: Let $a_1 = 1, a_2, \ldots, a_q$ be integers such that $\bar{a}_1, \ldots, \bar{a}_q$ are the distinct qth roots of unity in $\bar{Z}_p$. (They exist since $q \mid (p-1)$.) We define subdiagrams $\mathcal{Q}_i$, $0 \le i \le q$, as follows:

$$\mathcal{Q}_0: \quad Z \xrightarrow{f_0} \bar{Z}_q \xleftarrow{g_0} Z[\zeta_q], \qquad \text{where} \quad 1 \xleftarrow{g_0} \zeta_q$$

(Note that $\mathcal{Q}_0$ is just (3) of lemma 1.5, and $\ker(g_0) = \langle \zeta_q - 1 \rangle$, a maximal ideal of $Z[\zeta_q]$.)

$$\mathcal{Q}_1: \quad Z \xrightarrow{f_1} \bar{Z}_p \xleftarrow{g_1} \Lambda, \qquad \text{where} \quad \bar{a}_1 = 1 \xleftarrow{g_1} u_\sigma \quad \text{and} \quad 1 \xleftarrow{g_1} \zeta_p$$

For $2 \le i \le q$, define

$$\mathcal{Q}_i: \quad Z[\zeta_q] \xrightarrow{f_i} \bar{Z}_p \xleftarrow{g_i} \Lambda, \qquad \text{where} \quad 1 \xleftarrow{g_i} \zeta_p \quad \text{and} \quad \zeta_q \xrightarrow{f_i} \bar{a}_i \xleftarrow{g_i} u_\sigma$$

(Note that $\ker(f_i)$ equals $P_i = \langle \zeta_q - a_i, p \rangle$, a maximal ideal of $Z[\zeta_q]$, for $2 \le i \le q$, and in fact $pZ[\zeta_q] = P_2 \ldots P_q$, where $P_2, \ldots, P_q$ are distinct prime ideals. See Curtis and Reiner [2], section 4H for details.)

Theorem 1.7: ZG is isomorphic to the pullback of $Z \oplus Z[\zeta_q] \oplus \Lambda$ determined by the independent diagrams $\mathcal{Q}_0, \mathcal{Q}_1, \ldots, \mathcal{Q}_q$ defined above. Thus $ZG \cong \{(a,b,c) \in Z \oplus Z[\zeta_q] \oplus \Lambda \mid f_0(a) = g_0(b),\ f_1(a) = g_1(c),$ and $f_i(b) = g_i(c)$ for $2 \le i \le q\}$.

Proof: If we let $\bar{Z}_p[t]$ be the polynomial ring over $\bar{Z}_p$ in the indeterminate t, then $\bar{Z}_p[t]/\langle t^q - 1 \rangle \cong \bar{Z}_p H$ via the map induced by $t \longmapsto \sigma$. But $t^q - 1$ splits into linear factors in $\bar{Z}_p[t]$, so that using the Chinese remainder theorem, $\overline{Z}_pH \cong \overline{Z}_p[t]/<t^q-1> \cong \overline{Z}_p[t]/<t-\overline{a}_1> \oplus \cdots \oplus \overline{Z}_p[t]/<t-\overline{a}_q> \cong \overline{Z}_p \oplus \cdots \oplus \overline{Z}_p$, where $\overline{a}_1,\ldots,\overline{a}_q$ are as in notation 1.6.

Now let Φ_i be the map $\Phi: ZH \longrightarrow\!\!> \overline{Z}_pH$ (in (2)) followed by the projection to $\overline{Z}_p[t]/<t-\overline{a}_i>$ as above, and let Ψ_i be the map $\Psi:\Lambda \longrightarrow\!\!> \overline{Z}_pH$ followed by the projection to $\overline{Z}_p[t]/<t-\overline{a}_i>$. Note that $\Phi_i: ZH \longrightarrow\!\!> \overline{Z}_p$ is given by $\Phi_i(\sigma) = \overline{a}_i$ and $\Phi_i(n) = \overline{n} \in \overline{Z}_p$ for $n \in Z$, and $\Psi_i:\Lambda \longrightarrow\!\!> \overline{Z}_p$ is given by $\Psi_i(u_\sigma) = \overline{a}_i$, $\Psi_i(\zeta_p) = 1 \in \overline{Z}_p$, and $\Psi_i(n) = \overline{n} \in \overline{Z}_p$ for $n \in Z$. This allows us to split the diagram (2) into separate diagrams:

$$(4) \qquad ZH \xrightarrow{\Phi_i}\!\!> \overline{Z}_p <\!\!\xleftarrow{\Psi_i} \Lambda, \qquad \text{for} \quad 1 \leq i \leq q.$$

Given $(a,b) \in ZH \oplus \Lambda$, we have $\Phi(a) = \Psi(b)$ if and only if $\Phi_i(a) = \Psi_i(b)$ for $1 \leq i \leq q$, so we can restate lemma 1.4 as $ZG \cong \{(a,b) \in ZH \oplus \Lambda \mid \Phi_i(a) = \Psi_i(b)$ for $1 \leq i \leq q\}$, a multiple pullback.

Now consider ZH as a pullback of $Z \oplus Z[\zeta_q]$, as in lemma 1.5, identifying $m(\sigma) \in ZH$ with $(m(1),m(\zeta_q)) \in Z \oplus Z[\zeta_q]$. Then the map $\Phi_i: ZH \longrightarrow\!\!> \overline{Z}_p$ sends $m(\sigma)$ to $\overline{m(a_i)}$. When $i = 1$, a_1 is 1, so $\Phi_1(m(\sigma)) = \overline{m(1)} \in \overline{Z}_p$, which is simply the Z-coordinate of $m(\sigma) = (m(1),m(\zeta_q))$ reduced modulo p. Thus the image of $m(\sigma)$ under Φ_1 depends only upon its Z-coordinate, and we rewrite diagram (4), when $i = 1$, as $\mathcal{Q}_1$ of notation 1.6, where f_1 is Φ_1 restricted to Z, and g_1 is the map Ψ_1. Here f_1 is just the map "reduce modulo p".

When $2 \leq i \leq q$, we have $\overline{a}_i \neq 1$ is a primitive qth root of unity modulo p. In this case the map $f_i: Z[\zeta_q] \longrightarrow\!\!> \overline{Z}_p$ defined by $f_i(\zeta_q) = \overline{a}_i$ and $f_i(n) = \overline{n} \in \overline{Z}_p$ for $n \in Z$ is a well-defined ring homomorphism (with the maximal ideal $P_i = <\zeta_q - a_i, p>$ as kernel). Thus the image of $m(\sigma)$ under Φ_i depends only upon the image of $m(\zeta_q)$ under f_i, and we rewrite diagram (4) for $2 \leq i \leq q$ as $\mathcal{Q}_i$ of notation 1.6, where f_i is Φ_i restricted to $Z[\zeta_q]$, and g_i is the map Ψ_i.

It remains to show that subdiagrams $\mathcal{Q}_0,\mathcal{Q}_1,\ldots,\mathcal{Q}_q$ are independent. For Z we must show that the map $n \longmapsto (f_0(n),f_1(n)) \in \overline{Z}_q \oplus \overline{Z}_p$ for $n \in Z$ is onto, but this follows immediately from the Chinese remainder theorem, since the kernels of f_0 and f_1 are relatively prime. Likewise the Chinese remainder theorem implies that the map $\alpha \longmapsto (g_0(\alpha),f_2(\alpha),\ldots,f_q(\alpha)) \in \overline{Z}_q \oplus \overline{Z}_p^{(q-1)}$ for $\alpha \in Z[\zeta_q]$ is onto, since the kernels are relatively prime. (See the note in notation 1.6.) Finally, the map $\lambda \longmapsto (g_1(\lambda),\ldots,g_q(\lambda)) \in \overline{Z}_p^{(q)}$ for $\lambda \in \Lambda$ is onto, since this is just the map $\Psi:\Lambda \longrightarrow\!\!> \overline{Z}_pH$ of (2).□

2: MODULES OVER PULLBACKS

Now that we have represented ZG as a multiple pullback, the next step is to relate the structure of finitely generated modules over ZG to the structure of ZG as a pullback. This chapter establishes this translation by defining what will be called "ZG-diagrams", providing a functor from the category of ZG-diagrams to the category of ZG-modules, and showing that this functor is a representation equivalence. The result will be established for arbitrary modules and diagrams, not necessarily finitely generated, and then we shall show that, for ZG at least, finitely generated diagrams correspond to finitely generated modules. The results of this section are due to Levy (unpublished).

Since the results in this chapter hold for a larger class of rings than just ZG, and since the notation is actually easier in the general case, we introduce more general notation for this chapter. <u>We also assume, in this chapter only, that modules are not necessarily finitely generated.</u>

As in definition 1.1, let R be the pullback of $R_1 \oplus \dots \oplus R_m$ determined by a finite family of independent subdiagrams

$$\mathcal{Q}_k: \quad R_{i(k)} \xrightarrow{f_k} \!\!\!\!\to \overline{R}_k \leftarrow\!\!\!\!\!\xleftarrow{g_k} R_{j(k)}$$

for $1 \le k \le n$, where the coordinate rings R_c are arbitrary, and <u>we assume that the connecting rings $\overline{R}_k$ are semisimple artinian</u>. Set $\overline{R} = \bigoplus_{k=1}^{n} \overline{R}_k$. We first prove a simple lemma about R.

<u>Lemma 2.1</u>: The natural map $\nu: R \longrightarrow\!\!\!\!\to \overline{R}$ which sends $(r_1,\dots,r_m) \in R$ to $(\overline{r}_1,\dots,\overline{r}_n) \in \overline{R}$, where $\overline{r}_k = f_k(r_{i(k)}) = g_k(r_{j(k)})$, is onto. Its kernel is $\bigoplus_c (R \cap R_c)$, where $R \cap R_c$ denotes those $(r_1,\dots,r_m) \in R$ such that $r_b = 0$ when $b \neq c$.

<u>Proof</u>: That ν is onto follows immediately from the fact that the subdiagrams $\mathcal{Q}_1,\dots,\mathcal{Q}_n$ defining R are independent. Since $R \cap R_c \subseteq \ker(\nu)$, it follows that $\bigoplus_c (R \cap R_c) \subseteq \ker(\nu)$. Conversely, if $(r_1,\dots,r_m) \in \ker(\nu)$, then for each c, r_c maps to $0 \in \overline{R}_k$ in each subdiagram $\mathcal{Q}_k$ in which R_c occurs, so that $(0,\dots,0,r_c,0,\dots,0) \in R$. Thus $\ker(\nu) \subseteq \bigoplus_c (R \cap R_c)$.□

<u>Definition 2.2</u>: Let S_c be a left R_c-module for $1 \le c \le m$. An <u>R-diagram</u> $\mathcal{D}$ constructed from the coordinate modules $S_1,\ldots,S_m$ is defined to be a collection of commuting subdiagrams

$$\mathcal{D}_k: \quad \begin{array}{ccccc} S_{i(k)} & \xrightarrow{\ \gamma_k\ }\!\!\!\!\to & \overline{S}_k & \leftarrow\!\!\!\!\xleftarrow{\ \delta_k\ } & S_{j(k)} \\ & \nwarrow^{\alpha_k} & & {}^{\beta_k}\nearrow & \\ & & \overline{K}_k & & \end{array}$$

for $1 \le k \le n$, such that

(i) α_k, β_k, γ_k, and δ_k are R-homomorphisms, which is to say that α_k and γ_k are $R_{i(k)}$-homomorphisms, and β_k and δ_k are $R_{j(k)}$-homomorphisms;

(ii) γ_k and δ_k are onto, and α_k and β_k are one-to-one;

(iii) $\overline{S}_k$ and $\overline{K}_k$ are $\overline{R}_k$-modules; and

(iv) $\mathrm{im}(\alpha_k) \subseteq \ker(\gamma_k) = (\ker(f_k))S_{i(k)}$ and $\mathrm{im}(\beta_k) \subseteq \ker(\delta_k) = (\ker(g_k))S_{j(k)}$, where f_k and g_k are from the subdiagram $\mathcal{Q}_k$ in the definition of R.

Since the subdiagrams $\mathcal{Q}_1,\ldots,\mathcal{Q}_n$ are independent, it follows easily that the subdiagrams

$$\mathcal{D}'_k: \quad S_{i(k)} \xrightarrow{\ \gamma_k\ }\!\!\!\!\to \overline{S}_k \leftarrow\!\!\!\!\xleftarrow{\ \delta_k\ } S_{j(k)}$$

for $1 \le k \le n$, formed from $\mathcal{D}_k$ by ignoring the bottom half of the subdiagram, are independent (in the sense of definition 1.1). We let $S(\mathcal{D})$ denote the pullback of $S_1 \oplus \ldots \oplus S_m$ defined by these independent subdiagrams $\mathcal{D}'_1,\ldots,\mathcal{D}'_n$, that is, $S(\mathcal{D}) = \{(s_1,\ldots,s_m) \in S_1 \oplus \ldots \oplus S_m \mid \gamma_k(s_{i(k)}) = \delta_k(s_{j(k)}) \text{ for } 1 \le k \le n\}$.

Since the maps γ_k and δ_k are R-homomorphisms for each k, it follows that $S(\mathcal{D})$ is in fact an R-module. It is not the case, however, that every R-module can be represented in the form $S(\mathcal{D})$ for suitable $\mathcal{D}$. We shall use the modules $\overline{K}_k$ and the maps α_k and β_k to remedy this deficiency. First we require a few lemmas.

<u>Lemma 2.3</u>: Let $\mathcal{D}$ be an R-diagram. If we identify $\overline{K}_k$ with its image in $S_{i(k)} \oplus S_{j(k)}$ under the map $\lambda \longrightarrow (\alpha_k(\lambda),\beta_k(\lambda))$ for $\lambda \in \overline{K}_k$, then $\overline{K}_k \subseteq (R \cap R_{i(k)})S_{i(k)} \oplus (R \cap R_{j(k)})S_{j(k)}$.

Proof: By independence of the subdiagrams $\mathcal{Q}_1,\dots,\mathcal{Q}_n$ defining R, given k we can find $e_k \in R_{i(k)}$ such that $f_k(e_k) = 1 \in \overline{R}_k$ and e_k maps to 0 in all other $\overline{R}_{k'}$ to which $R_{i(k)}$ maps. It follows that

(1) $$e_k(\ker(f_k)) \subseteq R \cap R_{i(k)},$$

that is, every element of $e_k(\ker(f_k))$ can occur as coordinate $i(k)$ of an element of R whose other coordinates are all zero. On the other hand, since $\overline{K}_k$ is an $\overline{R}_k$-module and $f_k(e_k) = 1 \in \overline{R}_k$, multiplication by e_k is the identity map on $\overline{K}_k$. Thus using (1) we get $\alpha_k(\overline{K}_k) = \alpha_k(e_k\overline{K}_k) = e_k\alpha_k(\overline{K}_k) \subseteq e_k(\ker(f_k))S_{i(k)} \subseteq (R \cap R_{i(k)})S_{i(k)}$. A similar argument shows that $\beta_k(\overline{K}_k) \subseteq (R \cap R_{j(k)})S_{j(k)}$.□

Lemma 2.4: Let $\mathcal{D}$ be an R-diagram. If we identify $\overline{K}_k$ with its image in $S_{i(k)} \oplus S_{j(k)}$, then in fact $\overline{K}_k \subseteq S(\mathcal{D})$.

Proof: We show the stronger statement

(2) $$\overline{K}_k \subseteq (R \cap R_{i(k)})S(\mathcal{D}) \oplus (R \cap R_{j(k)})S(\mathcal{D}).$$

(Here we identify $R \cap R_c$ with $\{(r_1,\dots,r_m) \in R \mid r_b = 0 \text{ for } b \neq c\} \subseteq R$.) By independence of the subdiagrams $\mathcal{D}_1,\dots,\mathcal{D}_n$ defining $S(\mathcal{D})$, each projection map $S(\mathcal{D}) \twoheadrightarrow S_c$ is onto. Hence $(R \cap R_c)S(\mathcal{D}) = (R \cap R_c)S_c$, so lemma 2.3 implies $\overline{K}_k \subseteq (R \cap R_{i(k)})S(\mathcal{D}) \oplus (R \cap R_{j(k)})S(\mathcal{D})$.□

Now that we know that $\overline{K}_k \subseteq S(\mathcal{D})$ for each $1 \le k \le n$, we make the following definition.

Definition 2.5: Let $\mathcal{D}$ be an R-diagram built from coordinate modules $S_1,\dots,S_m$, and let $K(\mathcal{D}) = \sum_k \overline{K}_k$, computed inside $S(\mathcal{D})$. (By lemma 2.4 this makes sense.) Note that $K(\mathcal{D})$ is an R-module since the maps α_k and β_k are R-homomorphisms. We define

$$M(\mathcal{D}) = S(\mathcal{D})/K(\mathcal{D}),$$

so that $M(\mathcal{D})$ is also an R-module.

We shall show that the construction of $M(\mathcal{D})$ is general enough so that any R-module can be represented as $M(\mathcal{D})$ for suitable $\mathcal{D}$. First we make the collection of R-diagrams into a category.

Definition 2.6: Let R be fixed as above, and let $\mathcal{D}$ and $\tilde{\mathcal{D}}$ be R-diagrams built from coordinate modules $S_1,\dots,S_m$ and $\tilde{S}_1,\dots,\tilde{S}_m$, respectively. A morphism $\theta:\mathcal{D} \longrightarrow \tilde{\mathcal{D}}$ will be a collection of R-module homomorphisms

$\theta_c:S_c \longrightarrow \tilde{S}_c$, $\bar{\theta}_k:\bar{S}_k \longrightarrow \bar{\tilde{S}}_k$ and $\hat{\theta}_k:\bar{K}_k \longrightarrow \bar{\tilde{K}}_k$, where $1 \le c \le m$ and $1 \le k \le n$, such that for each k, the three-dimensional diagram consisting of $\theta_{i(k)}$, $\theta_{j(k)}$, $\bar{\theta}_k$, and $\hat{\theta}_k$ together with the subdiagrams $\mathcal{D}_k$ and $\tilde{\mathcal{D}}_k$ commutes. It is routine to verify that the class of R-diagrams with morphisms so defined forms a category.

We define the <u>direct sum</u> $\mathcal{D} \oplus \tilde{\mathcal{D}}$ of R-diagrams by taking the direct sum of modules and maps in each. Note that this agrees with the definition of direct sum in the category of R-diagrams. We say that $\mathcal{D}$ is <u>indecomposable</u> if and only if it is indecomposable in the category of diagrams. That is, $\mathcal{D}$ is indecomposable if and only if $\mathcal{D} \ne 0$ (that is, $S_c \ne 0$ for some c) and whenever $\mathcal{D} \cong \mathcal{D}_1 \oplus \mathcal{D}_2$, either $\mathcal{D}_1 = 0$ or $\mathcal{D}_2 = 0$.

Note that any morphism $\theta:\mathcal{D} \longrightarrow \tilde{\mathcal{D}}$ carries $K(\mathcal{D})$ to $K(\tilde{\mathcal{D}})$, so that θ canonically induces an R-module homomorphism $M(\theta):M(\mathcal{D}) \longrightarrow M(\tilde{\mathcal{D}})$. It is easy to check that $M(\ )$ is an additive functor from the category of R-diagrams to the category of R-modules. In the remainder of this chapter we show that $M(\ )$ is actually a representation equivalence. First we need to establish a few facts about diagrams.

<u>Lemma 2.7</u>: Let $S = S(\mathcal{D})$. Then $S \cap S_c = (R \cap R_c)S$ for $1 \le c \le m$.

<u>Proof</u>: Fix c, let $F_k = \ker(f_k)$ and $G_k = \ker(g_k)$, and let $I_h = (\bigcap_{k\ne h} F_k) \cap (\bigcap_{k\ne h} G_k)$, the intersections taken over all f_k which map R_c to $\bar{R}_k$ and over all g_k which map R_c to $\bar{R}_k$, except for $k = h$. By independence of the subdiagrams $\mathcal{Q}_1,\ldots,\mathcal{Q}_n$ we get

$$(3) \qquad R_c = \sum_h I_h,$$

the sum taken over all h such that R_c maps to $\bar{R}_h$ in $\mathcal{Q}_h$. (This follows easily from the fact that $F_h + I_h = R_c$ or $G_h + I_h = R_c$ for each h, depending on whether f_h or g_h maps R_c to $\bar{R}_h$.) It is also clear that

$$(4) \qquad R \cap R_c = (\bigcap_k F_k) \cap (\bigcap_k G_k),$$

the intersections taken over all f_k which map R_c to $\bar{R}_k$ and over all g_k which map R_c to $\bar{R}_k$, respectively. Hence since S is a pullback, we get

$$\begin{aligned} S \cap S_c &= (\bigcap_k \ker(\gamma_k)) \cap (\bigcap_k \ker(\delta_k)) \\ &= (\bigcap_k F_k S_c) \cap (\bigcap_k G_k S_c) \qquad \text{(by definition of } \mathcal{D}) \\ &= (\sum_k I_k)[(\bigcap_k F_k S_c) \cap (\bigcap_k G_k S_c)] \quad \text{(by (3))} \end{aligned}$$