Introduction to

LINEAR ALGEBRA

DONALD J. WRIGHT

C15/2

Introduction to

LINEAR ALGEBRA

DONALD J. WRIGHT
University of Cincinnati

WCB McGraw-Hill

Boston Burr Ridge, IL Dubuque, IA Madison, WI New York San Francisco St. Louis
Bangkok Bogotá Caracas Lisbon London Madrid
Mexico City Milan New Delhi Seoul Singapore Sydney Taipei Toronto

WCB/McGraw-Hill

*A Division of The **McGraw-Hill** Companies*

INTRODUCTION TO LINEAR ALGEBRA

This book is printed on acid-free paper.

1 2 3 4 5 6 7 8 9 0 QPF/QPF 9 3 2 1 0 9 8

ISBN 0-07-072098-3

Vice president and editorial director: *Kevin T. Kane*
Publisher: *JP Lenney*
Sponsoring editor: *Maggie Rogers*
Editorial assistant: *Amy Upgren*
Marketing manager: *Mary K. Kittell*
Project manager: *Marilyn M. Sulzer*
Senior production supervisor: *Mary E. Haas*
Designer: *Lorna Lo*
Supplement coordinator: *Stacy A. Patch*
Compositor: *Interactive Composition Corporation*
Typeface: *10.5/12 Times*
Printer: *Quebecor Printing Book Group/Fairfield, PA*

Library of Congress Cataloging-in-Publication Data

Wright, Donald J.
Introduction to linear algebra / Donald J. Wright. — 1st ed.
p. cm.
Includes index.
ISBN 0-07-072098-3
1. Algebra, Linear. I. Title.
QA184.W75 1999
512'.5—dc21 98-22676
CIP

www.mhhe.com

PREFACE

As a standard course in the second-year mathematics curriculum, Linear Algebra delivers material that is important to quite a variety of disciplines. But it also serves as a bridge between the intuitive and computational treatment usually found in freshman calculus and the more formal atmosphere of upper-division courses. For most students, the transition between these two modes of operation is something of a coming-of-age experience. They enter Linear Algebra unaccustomed to working with abstract concepts and inexperienced in communicating their thoughts in a clear and precise manner, yet their success in subsequent courses depends to a large extent on mastery of these skills. The challenge then is to develop their mathematical sophistication while exploring the landscape of linear algebra.

Abstraction is probably the most formidable feature on the horizon. In practice, it evolves from concrete experience. To appreciate the beauty and formalism of an axiomatically defined structure like a vector space, it is very helpful (maybe even necessary) to have experienced a concrete prototype in which the main ideas arise as natural responses to compelling real-life problems. Consequently, this book devotes its first six chapters to a thorough discussion of the basic facts of linear algebra in the context of Euclidean n-space. It begins with the task of describing the algebraic structure of the set of solutions of a system of linear equations, a problem that is both ubiquitous and compelling. Its solution involves a modest step into the realm of abstraction (adapting to the environment of n-space), but the ideas that arise (subspace, spanning set, linear independence, etc.) are firmly grounded in concrete three-dimensional experience, with ample support from geometry to help visualize the results. Formulating and exploring these concepts in higher dimensions is a substantive but manageable project for a second-year student. It provides plenty of opportunities to develop mathematical sophistication without resorting to a purely abstract structure that is sufficiently far removed from everyday experience as to seem foreign, threatening, or even irrelevant. With the foundation having been firmly established in n-space, the last chapter introduces the notion of an abstract vector space. At this point, the ideas surrounding linearity have become familiar. What remains is to point out their occurrence in other contexts and observe how the abstract structure serves as a unifying force that highlights the common features of different settings.

Chapter 1 discusses linear systems and the matrix algebra that is used to describe and manipulate them. Students find the computational aspects of this material straightforward, but when a proper response requires something more than a numerical calculation, they have difficulty expressing their insights in a comprehensible manner. Consequently, this chapter places a fair amount of emphasis on using notation to express oneself and on presenting one's thoughts in an organized sequence of steps that lead logically from the hypotheses to the conclusion.

The second chapter introduces Euclidean n-space as the context in which to study the solutions of a linear system. The fundamental notion of subspace arises from the properties of the solution set of a homogeneous linear system, and notions such as spanning set and linear independence are seen as natural attempts to describe those solutions in an efficient manner. A full discussion of the subspace structure unfolds, including coordinate systems and direct sum decompositions. The presentation draws heavily on the geometric interpretation of the ideas in 2- and 3-space. The level of sophistication is equal to that of customary treatments in abstract vector spaces, but this approach has the advantage that the practicality of the ideas is apparent from the start and the setting is not intimidating.

Orthogonality is the subject of Chapter 3. This topic occurs rather early in the text for several reasons. First, in exploring the advantages that acrue to working with a "rectangular" coordinate system, the big ideas developed in the previous chapter receive solid reinforcement and start to become part of the thinking process. Second, early treatment of orthogonal projection makes available an important geometric operation that is very useful in illustrating later concepts (such as linear transformations and eigenvalues). Finally, to demonstrate the use of the ideas of Chapter 2 with an important application, I wanted to include as early as possible the least-squares linear fit problem, and orthogonality plays a central role in that discussion.

Chapter 4 deals with linear transformations. The treatment places special emphasis on understanding geometrically how these functions rearrange the subspaces of 2- and 3-space, making considerable use of rotations, projections, and reflections to provide insight into concepts such as one-to-one, onto, and invertible. Matrix representations are discussed in detail, and in preparation for the later appearance of eigenvalues, a fair amount of the attention is given to utilizing coordinate systems with respect to which the action of the function is particularly easy to describe. The last section on isometries and similarities is a nice application of many previously developed ideas.

A fairly concise and calculational treatment of the determinant occupies Chapter 5. The concept is defined inductively by Laplace expansion, the object being to quickly produce the properties of the determinant for future use in finding eigenvalues. For those with the time and inclination, the last section explains the determinant's use in calculating the "volume" of an n-dimensional parallelepiped and in determining the orientation of a basis.

In Chapter 6, virtually all the concepts developed earlier come together in the analysis of linear operators on n-space. The notion of eigenvector arises from an attempt to find a coordinate system whose axes are invariant under the action of a given function. As in the earlier chapters, the presentation has a strong geometric flavor, and the treatment is therefore restricted to the case of real eigenvalues. It culminates in a discussion of symmetric matrices and the role orthogonal diagonalization plays in the analysis of quadratic forms.

The final chapter focuses on the process of capturing the essence of a particular concept via a set of axioms, which can then be used to extend the idea to a purely abstract setting. Guided by previous experience in n-space, it develops the abstract notions of vector space, norm, and inner product. Along the way, the student sees interesting manifestations of linearity in various contexts and discovers that at least algebraically all n-dimensional vector spaces are only superficially different from Euclidean n-space. The latter discussion emphasizes the use of coordinate isomorphisms to perform tasks in the abstract setting via familiar routines in n-space.

Throughout this book, the emphasis is on understanding and using concepts. The goal is to learn to think in terms of linear algebra notions such as linear combination and linear independence, and that sort of familiarity comes from using the ideas in a substantive way. That is not to say that the book is especially theoretical in tone. The exercise sets contain an ample number of routine problems designed to develop competence in standard calculational techniques, but they also include a rich assortment of problems whose solutions are one or two steps removed from a routine calculation. These latter exercises tend to be concrete (so they seem doable), but they force the student to engage the ideas to make the necessary connections. It is in wrestling with these exercises that the desired maturation starts to take place. Once students adjust to using the definitions and theorems as tools to solve such problems, they are well on their way to constructing proofs of general results.

The text contains a variety of examples and exercises that illustrate the usefulness of linear algebra, including a discussion of the adjacency matrix for a graph, analysis of predator-prey models, least-squares approximation, eigenvalue techniques for solving linear difference equations, and the classification of critical points of a function of two variables. These topics are not developed to any great depth, but sufficient exposure is provided to give the reader an appreciation of the many ways in which this subject connects with other disciplines.

A number of the exercises, particularly those involving applications, require the use of technology to carry out the computations. Most of the routine problems, however, are designed so as to yield readily to paper and pencil calculation. It is important to recognize that algorithms such as Gauss-Jordan elimination are not just means to produce answers. They are valuable ways of thinking about problems, and I believe one has to perform them a fair number of times by hand before a real appreciation of their conceptual and practical usefulness sinks in. On the other hand, once students demonstrate that they've digested a particular algorithm, I'm content to have them carry out the procedure with the push of a button. Ultimately, the decision as to when and how much to incorporate computational technology into the course depends on the taste and judgment of the instructor. The problem sets provide some opportunities to exercise both modes of operation.

In each exercise set some of the problems are marked with bold numerals or letters, indicating that the answer or, in some cases, a full solution can be found in the back of the book. A complete solutions manual is available to adopting instructors.

ACKNOWLEDGEMENTS

I am indebted to a number of my colleagues for patiently class testing early versions of this text. I would especially like to thank Joe Fisher, David Herron, David Minda, and Steve Pelikan for many helpful suggestions and discussions. I also appreciate the thoughtful comments of the following reviewers, many of whose recommendations are reflected in the final version of the book.

Robert Beezer, *University of Puget Sound*
Keith Chavey, *University of Wisconsin–River Falls*
Francis Conlan, *Santa Clara University*
Daniel Drucker, *Wayne State University*
Michael Ecker, *Pennsylvania State University*
Murray Eisenberg, *University of Massachusetts*
Sheldon Eisenberg, *University of Hartford*
John Ellison, *Grove City College*
S. B. Khleif, *Tennessee Tech University*
Patricia Johnson, *Ohio State University*
Christopher McCord, *University of Cincinnati*
David Meredith, *San Francisco State University*
Tom Metzger, *University of Pittsburgh*
David Weinberg, *Texas Tech University*

Finally, it has been a pleasure working with the editors and production staff at McGraw-Hill. I am particularly grateful to Jack Shira for his confidence in the early versions of this project and to Marilyn Sulzer for her careful attention to detail during production.

Donald J. Wright

CONTENTS

1

Linear Systems and Matrices

1.1 LINEAR EQUATIONS

A line in the Cartesian plane has equation

$$ax + by = c, \tag{1.1}$$

where a, b, and c are real numbers and at least one of a and b is not zero. That is to say, the line consists of the points in the plane whose Cartesian coordinates (x,y) satisfy (1.1). The analogous equation in three variables is

$$ax + by + cz = d, \tag{1.2}$$

and the points in space with Cartesian coordinates satisfying (1.2) constitute a plane (as long as one of a, b, and c is not zero). Equations (1.1) and (1.2) are the general linear equations in two and three variables, respectively. This notion of a linear equation can be extended to accommodate a larger number of variables, and in doing so it is helpful to establish some notation. The set of real numbers is denoted by $\mathcal{R}$. It is customary to use subscripts to distinguish notationally between four or more variables, so n distinct variables are indicated by writing $x_1, \ldots, x_n$. This convention also serves to order the variables, that is, x_1 is considered the first, x_2 the second, and so forth.

DEFINITION. Given a positive integer n, the general *linear equation* in $x_1, \ldots, x_n$ is

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where $a_1, \ldots, a_n, b \in \mathcal{R}$ and at least one of $a_1, \ldots, a_n$ is not zero.

Interest in linear equations stems from the fact that there are many practical problems in which these linear relationships are the appropriate means of expressing the interaction between the variables.

■ **EXAMPLE 1.1.** *Allocation of Resources.* A small business has a monthly advertising budget of B dollars, and the cost per ad in the local media is

1. Television . . . . a dollars
2. Radio . . . . b dollars
3. Newspaper . . . . c dollars
4. Neighborhood Newsletter . . . . d dollars.

Let x_1, x_2, x_3, and x_4 be the numbers of monthly ads placed in these media, respectively. Assuming the entire budget is used each month, the equation governing the allocation of monthly ads is

$$ax_1 + bx_2 + cx_3 + dx_4 = B. \ \square$$

■ **EXAMPLE 1.2.** *Networks.* Figure 1.1 is a schematic drawing of a hypothetical distribution network for some commodity, with arrows indicating the direction of flow through each branch of the network.

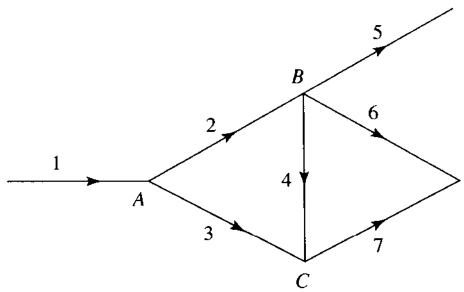


FIGURE 1.1

Suppose $x_1, \ldots, x_7$ are the amounts of the commodity distributed through branches $1, \ldots, 7$, respectively. Since the amount arriving at a given node is the same as the amount leaving that point, $x_1, \ldots, x_7$ are related as follows:

$$A: \quad x_2 + x_3 = x_1$$

$$B: \quad x_4 + x_5 + x_6 = x_2$$

$$C: \quad x_7 = x_3 + x_4.$$

Also, the amount arriving at A equals the total amount reaching the final destinations, so

$$x_1 = x_5 + x_6 + x_7.$$

The possible distribution scenarios are thus described by the linear equations

$$\begin{aligned} x_1 - x_2 - x_3 &= 0 \\ x_2 - x_4 - x_5 - x_6 &= 0 \\ x_3 + x_4 - x_7 &= 0 \\ x_1 - x_5 - x_6 - x_7 &= 0. \ \square \end{aligned}$$

■ **EXAMPLE 1.3.** *Interpolation of Data.* Consider the problem of fitting a given type of curve to experimentally obtained data points. For example, it seems likely that there is a parabola with a vertical axis that passes through (1,1), (2,3), and (3,−1). Such a curve is described by

$$y = a + bx + cx^2,$$

where $a, b, c \in \mathcal{R}$. That the curve is to pass through the given points is expressed by the following linear equations in the variables a, b, and c:

$$\begin{aligned} (1,1):& \quad 1 = a + b + c \\ (2,3):& \quad 3 = a + 2b + 4c \\ (3,-1):& \quad -1 = a + 3b + 9c. \ \square \end{aligned}$$

■ **EXAMPLE 1.4.** *Population Dynamics.* A given population is counted periodically and the kth census records n_k individuals, $k = 0, 1, \ldots$. A simple growth model assumes that the change in the population during the kth time period is proportional to the population size at the beginning of the period, that is, that $n_{k+1} - n_k = dn_k$, $d \in \mathcal{R}$. Then $n_{k+1} = (1 + d)n_k$, $k = 0, 1, \ldots$, so

$$n_k = (1 + d)n_{k-1} = (1 + d)^2 n_{k-2} = \cdots = (1 + d)^k n_0.$$

This model predicts exponential growth or decay, depending on whether d is positive or negative, respectively. A more interesting situation arises when two species interact, the classic case being the predator-prey relationship. Suppose f_k and r_k are census results for populations of foxes and rabbits, respectively. In the absence of rabbits, the fox population declines, say $f_{k+1} = \alpha f_k$, $0 < \alpha < 1$, and without foxes, the rabbit population expands, say $r_{k+1} = \delta r_k$, $\delta > 1$. If you assume that the presence of rabbits benefits the fox population in proportion to the number of rabbits and that foxes have a similar but negative influence on the rabbit count, then you get the following interaction model:

$$\begin{aligned} f_{k+1} &= \alpha f_k + \beta r_k \\ r_{k+1} &= -\gamma f_k + \delta r_k, \end{aligned}$$

where $\beta > 0$ and $\gamma > 0$. Note that the two equations are linear in r_k and f_k. □

A *solution* to

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b \tag{1.3}$$

is an ordered set of n real numbers, which, when substituted for their associated variables, satisfy the equation. An ordered set of n real numbers is called an *n-tuple*, written $(x_1, \ldots, x_n)$. As long as one of $a_1, \ldots, a_n$ is not zero, (1.3) has a solution. For example, if $a_1 \neq 0$, then (1.3) can be solved for x_1 to obtain

$$x_1 = (1/a_1)\{b - a_2 x_2 - \cdots - a_n x_n\}.$$

Any choice of values for $x_2, \ldots, x_n$ generates a corresponding value for x_1, and the resulting n-tuple is a solution. If $a_i \neq 0$ for some $i \in \{1, \ldots, n\}$, then a similar expression can be obtained for x_i. In that event, the variables other than x_i can be assigned arbitrary values, and they in turn determine x_i. There are, in fact, infinitely many solutions (provided $n \geq 2$). Each description of the solutions involves assigning values independently to $n - 1$ of the variables, so the general solution is said to depend on $n - 1$ parameters.

■ **EXAMPLE 1.5.** Solving

$$-x + 2y - 3z = 6$$

for x yields $x = -6 + 2y - 3z$. If y and z are assigned arbitrary values s and t, respectively, then the set of solutions is

$$\{(-6 + 2s - 3t, s, t) \ : \ s, t \in \mathcal{R}\}.$$

Alternatively, setting $x = s$ and $z = t$ and solving for y produces

$$\{(s, 3 + s/2 + 3t/2, t) \ : \ s,t \in \mathcal{R}\}.$$

In either event, the solutions are described in terms of two parameters, s and t. A third description, namely

$$\{(s, t, -2 - s/3 + 2t/3) \ : \ s,t \in \mathcal{R}\},$$

results from setting $x = s$ and $y = t$ and solving for z. □

Exercises 1.1

1. Which of the following equations are linear in the variables x, y, and z?
   a. $2x - 3y = \ln(2)$
   b. $5z - \frac{1}{2}x = 3y + 2$
   c. $\pi x + \sqrt{2}y + \frac{z}{e} = 0$
   d. $2xy + 3z = -1$
   e. $2\cos(x) + 3\sin(y) = \pi$
   f. $x\cos(2) + y\sin(3) = \pi$
   g. $x + \frac{2}{y} = 3$
   h. $z - 4 = \frac{x + 3y}{2}$
   i. $\sqrt{x} - \sqrt{y} = 1$
   j. $3y - 5 = 2\frac{x}{z}$
   k. $z = 4$
   l. $y - 1 = 3z^2 - 2x$

2. Interpreting each as a linear equation in x, y, and z, describe the solution set.
   a. $3x + 2y - z = 0$
   b. $2 - x + 3z = 4y$
   c. $x - \frac{1}{2}y = z + 2$
   d. $2y - z = 1$
   e. $x + 2z = 0$
   f. $2x = 3y$
   g. $3y = 4$
   h. $4 + 3x = -1$

3. A cash box containing p pennies, n nickels, d dimes, and q quarters has a total of \$37.65 in coins. Find an equation relating the values of p, n, d, and q. Is the equation linear?

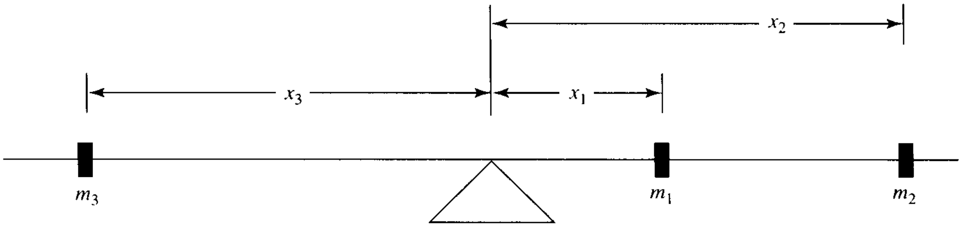
4. A parabola with a horizontal axis is to be fitted to the data points in Example 1.3. State the general equation of such a curve and find the conditions the coefficients must satisfy if the curve is to pass through the given points.

5. The general gas law states that the pressure P, volume V, and temperature T, of a sample consisting of n moles of an ideal gas are related by

$$PV = n(8.317)T. \qquad (*)$$

   a. Is $(*)$ a linear equation in P, V, and T?
   b. When T is held constant, $(*)$ is known as Boyle's law. Is Boyle's law a linear equation in P and V?
   c. When P is held constant, $(*)$ is known as Charles's law. Is Charles's law a linear equation in V and T?

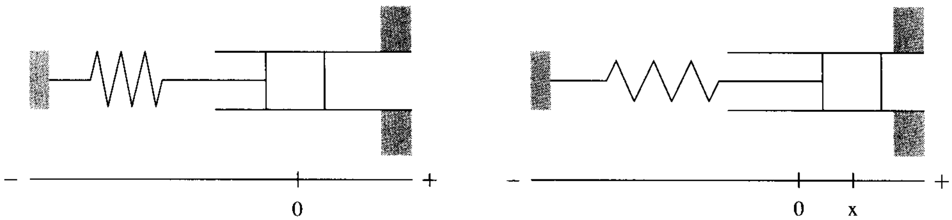
6. Masses m_1, m_2, and m_3 are located on a balance beam at distances x_1, x_2, and x_3, respectively, from the fulcrum (as illustrated in the figure).



What condition must be satisfied for this system of masses to be in equilibrium (i.e., for the beam to balance)? Is this condition a linear equation in the variables

a. m_1, m_2, and m_3? **b.** x_1, x_2, and x_3? c. m_1, m_2, m_3, x_1, x_2, and x_3?

7. Consider a mechanical system that consists of a spring connecting a fixed support to a piston that slides back and forth in a stationary cylinder. Figure (a) shows the system in equilibrium, with 0 indicating the rest position of the piston. When displaced and released, the piston exhibits an oscillatory motion with position x, velocity v, and acceleration a. Hooke's law states that the spring exerts a restoring force on the piston opposite in direction to the displacement with magnitude proportional to the size of the displacement. The constant of proportionality, called the spring constant, is denoted by $k > 0$. The cylinder produces a damping effect by exerting a force on the piston directed opposite to the direction of motion, with magnitude proportional to the magnitude of the velocity. Assume $d > 0$ is the constant of proportionality. By Newton's second law of motion, the sum of the forces acting on the piston is equal to the product of its mass m and its acceleration.



(a) (b)

a. Find the equation governing the motion of the piston.
b. Is the answer in part (a) a linear equation in x, v, and a?

1.2
LINEAR SYSTEMS

Given positive integers m and n, a collection of m linear equations in the same n variables is called a *linear system*. A *solution* for such a system is an n-tuple of real numbers that satisfies all the equations. Each equation generally has infinitely many solutions, but there may not be any n-tuples that satisfy all m equations, so the first issue regarding linear systems is the question of existence of solutions. The second, given that there are solutions, is the problem of identifying and describing them. When the system involves only two or three variables, these issues have simple geometric interpretations.

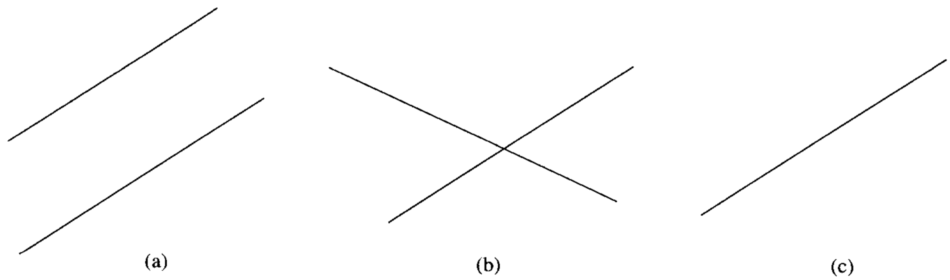


FIGURE 1.2

When $n = 2$, each equation represents a line in the plane, and a solution is a point that is common to all of the lines. For a system of two equations ($m = 2$), there are three possible outcomes, as illustrated in Figure 1.2. The two lines may be distinct but parallel (Figure 1.2(a)), in which case the system has no solutions. Alternatively, the lines intersect and, therefore, either meet in a single point (Figure 1.2(b)) or are identical (Figure 1.2(c)). Of the latter two possibilities, the first corresponds to a system with a unique solution, the second to a system with infinitely many solutions that constitute a line.

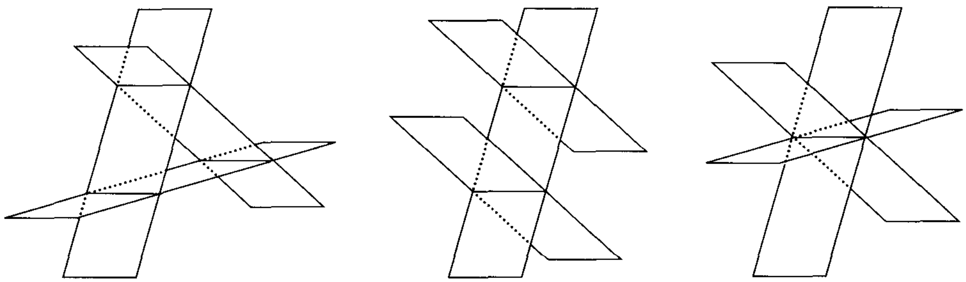
When the system has three or more equations, it is still the case that the corresponding lines either have no points in common, have a single point in common, or are all the same. In general then, a system of m equations in two unknowns has either no solution, a unique solution, or infinitely many solutions that form a line.

When $n = 3$, each equation represents a plane in space. Suppose first that there are two equations. If they represent the same plane, then the solutions of the system are just the points of that plane. Alternatively, they represent distinct planes that either intersect in a line or are parallel. The first of the latter options corresponds to a system with infinitely many solutions (that constitute a line), the second corresponds to a system with no solutions. Note that having a unique solution is not one of the possibilities.

Now suppose the system has three equations. The case when two or more of them represent the same plane is covered by the previous discussion, so assume the planes are distinct. If they are all parallel, then the system has no solution. Otherwise, at least two of them meet. In that event, the remaining plane either is or is not parallel to the line of intersection of the two. The first of these possibilities can occur in three essentially different ways, as illustrated in Figure 1.3. Two of them (Figures 1.3(a) and 1.3(b)) are situations in which there are no solutions, and the third (Figure 1.3(c)) corresponds to a system whose solutions form a line. Finally, when two of the planes meet and the line of intersection is not parallel to the third plane, that line intersects the third plane in a single point. In this case, the system has a unique solution.

Regardless of the number of equations in the system, there are always only four possible outcomes: the planes may have nothing in common, a point in common, a line in common, or a plane in common.

The foregoing discussion indicates that when $n = 2$ or $n = 3$, the solution set comes in one of a few very special forms (a point, a line, a plane, or the empty set). For



(a) (b) (c)

FIGURE 1.3

larger values of n, the equations in the system no longer represent familiar geometric objects, but the solution set still has a certain form that can be characterized algebraically. The concepts that have been developed to describe the "algebraic structure" of such a set make up the core of the subject known as *linear algebra*. The present study of that subject begins therefore by examining algebraic techniques for solving linear systems.

A general linear system of m equations in n unknowns looks like

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

The real numbers, a_{ij}, $1 \le i \le m$, $1 \le j \le n$, are called the *coefficients* of the system. Note that the first index in a_{ij} indicates the equation in which the coefficient occurs, whereas the second identifies the variable that it multiplies. The numbers $b_1, \ldots, b_n$ are called the *constant terms*. A system that has a solution is said to be *consistent*, whereas one that does not is called *inconsistent*.

The strategy for finding solutions is to replace the given system by another system that has the same solutions but whose form is sufficiently simple that the solutions are apparent. The simpler system is obtained by repeating one basic process. Solve one equation for one of the variables, producing an expression for it in terms of the others. Then substitute that expression in the other equations. The effect is to eliminate the one variable from the other equations. The new system consists of a single equation in n variables and some replacement equations involving only $n - 1$ variables.

■ **EXAMPLE 1.6.** Consider

$$\begin{aligned} 2x_1 + 3x_2 - x_3 &= 3 \\ x_1 - x_2 + 2x_3 &= -1. \end{aligned}$$

The second equation gives $x_1 = -1 + x_2 - 2x_3$, and substituting in the first equation yields

$$2(-1 + x_2 - 2x_3) + 3x_2 - x_3 = 3,$$

or

$$5x_2 - 5x_3 = 5.$$

The last line is further simplified by dividing both sides by 5, whereupon the original system has been replaced by

$$\begin{aligned} x_1 - x_2 + 2x_3 &= -1 \\ x_2 - \ \ x_3 &= \ \ 1. \end{aligned}$$

In the latter system, each choice of x_3 in the second equation determines a value of x_2, and the values of x_2 and x_3, together with the first equation, determine x_1. There are infinitely many solutions depending on one parameter. Setting $x_3 = t$ gives $x_2 = 1 + t$ and $x_1 = -1 + x_2 - 2x_3 = -1 + (1 + t) - 2t = -t$. The solutions are the 3-tuples $(-t, 1 + t, t), t \in \mathcal{R}$. In this example, the two equations represent planes in space, and the solution set is the line of intersection of those planes. It is described parametrically by

$$x_1 = -t, \qquad x_2 = 1 + t, \qquad x_3 = t, \qquad t \in \mathcal{R}. \ \square$$

■ **EXAMPLE 1.7.** Recall the system generated in Example 1.3, that is,

$$\begin{aligned} a + \ \ b + \ \ c &= \ \ 1 \\ a + 2b + 4c &= \ \ 3 \\ a + 3b + 9c &= -1. \end{aligned}$$

From the first equation you have $a = 1 - b - c$, which, when substituted in the other equations, produces

$$\begin{aligned} b + 3c &= \ \ 2 \\ 2b + 8c &= -2. \end{aligned}$$

Multiplying through the last line by 1/2, the new system becomes

$$\begin{aligned} a + b + \ \ c &= \ \ 1 \\ b + 3c &= \ \ 2 \\ b + 4c &= -1. \end{aligned}$$

Note that the variable a no longer occurs in the last two equations. Now repeat the process, using the second equation of the new system to eliminate b from the third equation. The result is

$$\begin{aligned} a + b + \ \ c &= \ \ 1 \\ b + 3c &= \ \ 2 \\ c &= -3. \end{aligned}$$

Since $c = -3$, the second equation gives $b = 11$, and the first equation then yields $a = -7$. The system has a unique solution, $(-7, 11, -3)$. Referring again to Example 1.3, this conclusion confirms that there is a unique parabola with a vertical axis that passes through (1,1), (2,3), and $(3,-1)$. Its equation is $y = -7 + 11x - 3x^2$. □

■ **EXAMPLE 1.8.** Suppose you attempt to fit a line to the points in Example 1.3. Since the three points are noncollinear, you cannot succeed, and the impossibility of the task should become algebraically apparent while trying to solve the resulting system of equations. The proposed line has equation $y = mx + b$, and (1,1), (2,3), and (3,−1) lie on the line if and