

# Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

1329

M. Alfaro J.S. Dehesa F.J. Marcellan  
J.L. Rubio de Francia J. Vinuesa (Eds.)

## Orthogonal Polynomials and their Applications

Proceedings, Segovia 1986



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## Orthogonal Polynomials and their Applications

Proceedings of an International Symposium  
held in Segovia, Spain, Sept. 22–27, 1986

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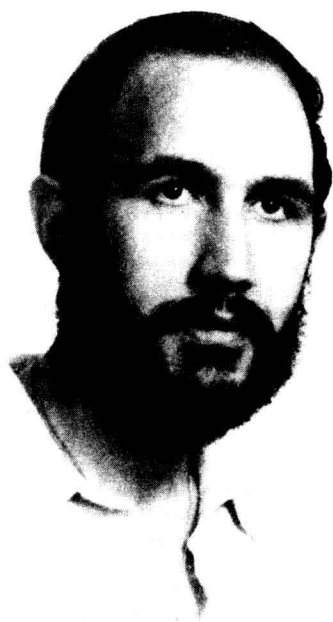
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This volume is dedicated to the memory of Jose Luis Rubio de Francia, Professor of Mathematics of the Autonomous University of Madrid, who died on February 1988. His death is a great loss to his country, to science and education, and to his many friends. We wish to record here our profound sense of loss. We hope that the completed volume is faithful to his vision of a book that not only instructs but also inspires.



## INTRODUCTION

The present volume contains the Proceedings of the Second International Symposium on Orthogonal Polynomials and Their Applications held in Segovia, Spain, September 22<sup>nd</sup> to 27<sup>th</sup>, 1986.

This Symposium continues the idea born at Bar-le-Duc (France) on the occasion of the 150th anniversary of Edmond Nicolas Laguerre's birthday.

It intends to present a comprehensive view of the main achievements, fundamentals and working problems of the orthogonal polynomials and its relationship with other mathematical fields and applied sciences.

The core of the Symposium was a series of ten invited lectures, each covering some of the more fundamental aspects. In addition, there were 60 half-hour communications, including particular aspects of orthogonal polynomials on the real line and over the complex plane, special functions, orthogonal expansions and numerical methods. A special session devoted to open problems in the field of orthogonal polynomials also took place.

This volume contains nine invited lectures, more than ten seminars and a collection of some open questions.

Many institutions and individuals helped make the Symposium possible. We particularly thank, for their financial support, the Comisión Asesora de Investigación Científica y Técnica (CAICYT), Consejo Superior de Investigaciones Científicas (CSIC), Confederación Española de Centros de Investigación Matemática y Estadística (CECIME), Universidad Politécnica de Madrid, IBM-España S.A., Diputación Provincial de Segovia, Ayuntamiento de Segovia y Caja de Ahorros y Monte de Piedad de Segovia. For the hospitality offered to the participants, we thank the Ayuntamiento and Diputación Provincial de Segovia. Finally it is a pleasure to acknowledge Mónica Garay and Russell di Napoli for their secretarial help. We are also greatly indebted to the series editors of the Lecture Notes in Mathematics for giving us the possibility to publish these Proceedings in this series.

M. Alfaro  
J. S. Dehesa  
F. Marcellán

J. L. Rubio  
J. Vinuesa

- DRAUX, A.: On quasi-orthogonal polynomials.
- EIERMANN, M.: Orthogonal polynomials and semiiterative methods for nonsymmetric systems of linear equations.
- GAUTSCHI, W., LANDAU, H.J. and MILOVANOVIC, G.V.: Polynomials on the semicircle II.
- GONZALEZ, C.: Convergence of some matrix Padé-type approximants to the exponential.
- GORI, L.: On the behaviour of the zeros of some s-orthogonal polynomials.
- HENDRIKSEN, E.: A characterization of classical orthogonal Laurent polynomials.
- KURATSUBO, S.: A note on the summability of the Riesz means of multiple Fourier series.
- LASCOUX, A.: Interpolation de Lagrange.
- LUBINSKY, D.S. and SAFF, E.B.: Approximation by exponentially weighted polynomials with applications to asymptotics for orthogonal polynomials.
- LUQUIN, F.: Approximation by polynomials with integral coefficients.
- MARKETT, C.: A convolution structure for Fourier-Bessel series and the harmonic analysis of singular eigenfunction expansions.
- MEHTA, M.L.: Some identities for Hermite polynomials and the discrete Fourier Transform.
- MEIJER, H.G., BAVINCK, H.: An ultraspherical generalization of Keall's Jacobi type polynomials.
- NAMASIVAYAM, S., ORTIZ, E.L.: Some remarks on the structure of the error of tau methods.
- OSOWSKI, S.: Explicit formulas for the sensitivity of tridiagonal systems.
- PEREZ-ACOSTA, F.: Adaptive orthogonal collocation.
- PEREZ-GRASA, I.: Kernels and polynomial modifications in lemniscates.
- PIÑAR, M., RAMIREZ, V.: Recursive inversion of Hankel matrices.

- POPOV, B.: A class of polynomials connected with orthogonal polynomials.
- RONVEAUX, A.: Semi-classical weights  $(-\infty, +\infty)$  semi-hermite orthogonal polynomials.
- SABLONNIERE, P.: Hahn polynomials as eigenvectors of positive operators.
- SANSIGRE, G., FERNANDEZ, L., ALVAREZ, M.: On zeros of general orthogonal polynomials on the unit circle.
- SMAILI, N.: The class-T discrete semi-classical polynomials.
- SOTTAS, G.: Positivity domain of ultraspherical type quadrature formulas with Jacobi abscissas: Numerical investigations.
- SZAFRANIEC, F.H.: Orthogonal polynomials and subnormality of related shift operators.
- TEMME, N.M.: Laguerre Polynomials: Asymptotics for Large Degree.
- VAN DOORN, E.A.: Orthogonal polynomials, chain sequences and oscillation criteria for second order linear difference equations.
- VAN ISEGHEM, J.: Zeros of vector-orthogonal polynomials.
- WALLIN, H.: Orthogonal polynomials on fractals.
- ZAYED, A.: A generalized inversion formula for the continuous Jacobi Transform.

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## TABLE OF CONTENTS

|                                                                                                                                                                      |     |
|----------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----|
| Introduction . . . . .                                                                                                                                               | V   |
| List of Participants . . . . .                                                                                                                                       | XI  |
| <br><u>I. INVITED SPEAKERS</u>                                                                                                                                       |     |
| BREZINSKI, C.: Error estimate in Padé Approximation . . . . .                                                                                                        | 1   |
| EDELMAN, A., MICCHELLI, C.A.: Orthogonal Polynomials in monotone and convex interpolation . . . . .                                                                  | 20  |
| GERONIMO, J.S.: Polynomials orthogonal with respect to singular continuous measures . . . . .                                                                        | 32  |
| KOORNWINDER, T.H.: Group theoretical interpretation of Askey's scheme of hypergeometric orthogonal polynomials . . . . .                                             | 46  |
| KRALL, A.M.: A review of orthogonal polynomials satisfying boundary value problems . . . . .                                                                         | 73  |
| LITTLEJOHN, L.L.: Orthogonal polynomial solution to ordinary and partial differential equations . . . . .                                                            | 98  |
| LOPEZ, G., RAKHMANOV, E.A.: Rational approximation orthogonal polynomials and equilibrium distributions . . . . .                                                    | 125 |
| MATE, A., NEVAI, P.: Factorization of second order difference equations and its applications to orthogonal polynomials . . . . .                                     | 158 |
| RIVLIN, T.J.: Gauss quadrature for analytic functions . . . . .                                                                                                      | 178 |
| <br><u>II. CONTRIBUTORS</u>                                                                                                                                          |     |
| AL SALAM, W.A., VERMA, A.: On the Geronimus polynomial sets . . . . .                                                                                                | 193 |
| BARATELLA, P., GATTESCHI, L.: The bounds for the error term of an asymptotic approximation of Jacobi polynomials . . . . .                                           | 203 |
| BUENDIA, E., S. DEHESA, J., GALVEZ, F.J.: The distribution of zeros of the polynomial eigenfunctions of ordinary differential operators of arbitrary order . . . . . | 222 |
| CACHAFEIRO, M.A., MARCELLAN, F.: Orthogonal polynomials and jump modifications . . . . .                                                                             | 236 |

|                                                                                                                                |     |
|--------------------------------------------------------------------------------------------------------------------------------|-----|
| HENDRIKSEN, E., VAN ROSSUM, H.: Electrostatic interpretation of zeros . . . . .                                                | 241 |
| ISMAIL, M.E.H., LETESSIER, J., VALENT, G.: Associated dual Hahn polynomials. . . . .                                           | 251 |
| LOPEZ, G.: Szegő's Theorem for polynomials orthogonal with respect to varying measures . . . . .                               | 255 |
| MAGNUS, A.: Associated Askey-Wilson polynomials as Laguerre Hahn orthogonal polynomials . . . . .                              | 261 |
| MARONI, P.: Le calcul des formes linéaires et les polynômes orthogonaux semi-classiques . . . . .                              | 279 |
| MASON, J.C.: The $L_p$ minimality and near-minimality of orthogonal polynomial approximation and integration methods . . . . . | 291 |
| NJASTAD, O.: Orthogonal rational functions with poles in a finite subset of $\mathbb{R}$ . . . . .                             | 300 |
| STAHL, H.: $N^{\text{th}}$ root asymptotics for orthogonal polynomials                                                         | 308 |
| TORRANO, E., GUADALUPE, R.: Zeros of orthogonal polynomials on harmonic algebraic curves . . . . .                             | 320 |
| III. <u>OPEN PROBLEMS</u> . . . . .                                                                                            | 328 |
| IV. <u>COMMUNICATIONS NOT PUBLISHED IN THIS VOLUME</u>                                                                         |     |
| ALFARO, M.P., VIGIL, L.: On extensions of finite sequences of orthogonal polynomials.                                          |     |
| ANTOLIN, J.: Inverse problems, moment theory and the maximum entropy formalism.                                                |     |
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| BACRY, H., BOON, M.: Hardy-Pollaczek polynomials and $\Gamma$ and $\zeta$ functions.                                           |     |
| BELMEHDI, S.: Une caractérisation des polynômes orthogonaux semi-classiques.                                                   |     |
| BOUKHEMIS, A.: Une caractérisation des polynômes strictement $1/p$ -orthogonaux de type Sheffer. Etude du cas $p = 2$ .        |     |
| CASASUS, L., GONZALEZ, P.: Orthogonal polynomials and geometric convergence of Padé type approximants.                         |     |

# ERROR ESTIMATE IN PADÉ APPROXIMATION

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## ABSTRACT.

The aim of this paper is to extend Kronrod's procedure for estimating the error in Gaussian quadrature formulas to Padé approximation. Some results on Stieltjes polynomials are also given. Examples show the effectiveness of the method. The procedure is then applied to the  $\epsilon$ -algorithm, which is a convergence acceleration method related to Padé approximation. General principles for estimating the error in series approximations and sequence transformations are also brought to light.

## INTRODUCTION.

In 1965, A.S. Kronrod published a small book [10] in which he proposed an heuristic method for estimating the error in Gaussian quadratures. It simply consists in comparing the results obtained by two different quadrature formulas. If, for example, one makes use of two Gaussian quadratures with respectively  $n$  and  $n+1$  nodes then one has to evaluate the integrand at  $2n+1$  points. This procedure is not the best since a Gaussian quadrature with  $n+1$  nodes exactly integrates polynomials up to the degree  $2n+1$  while we are using  $2n+1$  points for the same accuracy. We are wasting the integrand evaluations. Thus, Kronrod's idea was to use a Gaussian quadrature formula first and then to add it new nodes in an optimal way that is in order to obtain a new formula exact for polynomials of the highest possible degree. The difference between the two results gives an estimate of the error.

Since Padé approximants can be viewed as formal Gaussian quadratures [2], the aim of this paper is to extend Kronrod's idea to Padé approximation.

Let us recall some results. Let  $f$  be a formal power series

$$f(t) = \sum_{i=0}^{\infty} c_i t^i, \quad c_i \in \mathcal{C}$$

Let  $c$  be the linear functional acting on the space of complex polynomials, defined by

$$c(x^i) = c_i \quad i = 0, 1, \dots$$

We formally have

$$f(t) = c((1-xt)^{-1}).$$

Let  $v$  be an arbitrary polynomial of degree  $k$ . We define  $w$  as

$$w(t) = c\left(\frac{v(x)-v(t)}{x-t}\right)$$

where  $c$  acts on the variable  $x$ ,  $t$  being a parameter.  $w$  is a polynomial of degree  $k-1$ .  $w$  is the polynomial associated to  $v$ . We set

$$\tilde{v}(t) = t^k v(t^{-1}) \quad \text{and} \quad \tilde{w}(t) = t^{k-1} w(t^{-1}).$$

The rational fraction  $\tilde{w}(t)/\tilde{v}(t)$  is called a Padé-type approximant of  $f$ . It is denoted by  $(k-1/k)_f(t)$ .  $v$  is called its generating polynomial. We have

$$f(t) - (k-1/k)_f(t) = \frac{t^k}{\tilde{v}(t)} c\left(\frac{v(x)}{1-xt}\right) = O(t^k).$$

Padé-type approximants can be related to polynomial interpolation. Let  $P$  be the Hermite interpolation polynomial of  $(1-xt)^{-1}$  (as a function of  $x$ ,  $t$  being a parameter) at the zeros of  $v$ . We have

$$(k-1/k)_f(t) = c(P(x)).$$

Thus  $(k-1/k)$  can be viewed as a formal interpolatory quadrature formula for  $(1-xt)^{-1}$  with the zeros of  $v$  as nodes. The above error expression is just the classical property of an interpolatory quadrature formula with  $k$  nodes to be exact for polynomials of degree at most  $k-1$ . Moreover, from that expression, we have

$$f(t) - (k-1/k)_f(t) = \frac{t^k}{\tilde{v}(t)} \{c(v(x)) + c(xv(x))t + \dots + c(x^{k-1}v(x))t^{k-1} + t^k c\left(\frac{x^k v(x)}{1-xt}\right)\}$$

Since  $v$  can be arbitrarily chosen, let us take it such that

$$c(x^i v(x)) = 0 \quad i = 0, \dots, k-1.$$

This is always possible if the Hankel determinant

$$H_k(0) = \begin{vmatrix} c_0 & c_1 & \dots & c_{k-1} \\ c_1 & c_2 & \dots & c_k \\ \dots & \dots & \dots & \dots \\ c_{k-1} & c_k & \dots & c_{2k-2} \end{vmatrix}$$