Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

Subseries: Instituto de Matemática Pura e Aplicada, Rio de Janeiro

Adviser: C. Camacho

1207

Pierre H. Bérard

Spectral Geometry: Direct and Inverse Problems

With an Appendix by G. Besson

Springer-Verlag

Berlin Heidelberg New York London Paris Tokyo

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This book is being published in a parallel edition by the Instituto de Matemática Pura e Aplicada, Rio de Janeiro as volume 41 of the series „Monografías de Matemática".

Mathematics Subject Classification (1980): Primary: 58G25, 35P15, 52A40
Secondary: 58G11, 58C40, 58G30

ISBN 3-540-16788-9 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-16788-9 Springer-Verlag New York Berlin Heidelberg

Library of Congress Cataloging-in-Publication Data. Bérard, Pierre H. Spectral geometry. (Lecture notes in mathematics; 1207) Bibliography: p. Includes index. 1. Geometry, Riemannian. 2. Eigenvalues. 3. Operator theory. I. Title. II. Series: Lecture notes in mathematics (Springer-Verlag); 1207.
QA3.L28 no. 1207 [QA649] 510 s [516.3'73] 86-20323
ISBN 0-387-16788-9 (U.S.)

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Printed in Germany

Printing and binding: Druckhaus Beltz, Hemsbach/Bergstr.
2146/3140-543210

To Rachel, Philippe, Izabel

INTRODUCTION

The purpose of these notes is to describe some aspects of direct problems in spectral geometry.

Eigenvalue problems were motivated by questions in mathematical physics. In these notes, we deal with eigenvalue problems for the Laplace-Beltrami operator on a compact Riemannian manifold. To such a manifold (M,g), we can associate a sequence of non-negative real numbers $\{\lambda_i\}_{i\geq 1}$, the eigenvalues of the Laplace-Beltrami operator Δ^g acting on $C^\infty(M)$. One can think of a Riemannian manifold as a musical instrument together with the musician who plays it. In this picture, the eigenvalues of the Laplace operator correspond to the harmonics of the instrument; they may depend on the music player, i.e. on the Riemannian metric: think of a kettledrum, or better of a Brazilian "cuíca".

Spectral geometry aims at describing the relationships between the musical instrument and the sounds it is capable of sending out.

The problems which arise in spectral geometry are of two kinds: direct problems and inverse problems. In a direct problem, we want information on the sounds produced by the instrument, in terms of its geometry. For example, we know that the bigger the tension of the parchment head of a kettledrum, the higher the pitch. In an inverse problem, we investigate what geometric information on the instrument can be recovered from the sounds it sends out.

Both types of problems are relevant to deep questions arising in mathematical physics (for example in elasticity theory, in plasma physics, in spectroscopy...).

This book could be divided into three parts: Chapters I to III; Chapters IV to VI and Appendix A; Chapter VII and Appendices B and C.

In Chapter I, we give some very simple-minded motivations from mathematical physics. Our purpose is not to derive mathematical models for some physical phenomena, but rather to show how some mathematical objects which will be introduced later on, arise naturally from physical principles. For further reading, we suggest [C-H] and [TL].

Chapter II is devoted to Riemannian geometry. We introduce the basic notions (geodesics, curvature,...) and we state, mainly without proofs, the basic results. In order to understand Chapter VI, the reader must have in mind the comparison theorems which involve the curvature of a Riemannian manifold. For further reading, we suggest [B-C], [CO], [C-E], [M-S] and [SI].

In Chapter III, we introduce the Laplace-Beltrami operator, and we describe the eigenvalue problems we will deal with in this book. An important part of this chapter is devoted to the variational characterizations of the eigenvalues. This is very important for later purposes. Although this material can be considered as classical ([KO], [R-S] or [C-H]), we have tried to describe it at length. The last paragraph of Chapter III contains general considerations on direct and inverse problems, and some answers to such problems as an illustration of the variational characterizations of the eigenvalues.

Chapters IV to VI form the core of this book. They contain results related to isoperimetric inequalities and to an important topic in Riemannian geometry, namely the interactions between local geometry (curvature estimates) and global geometry (topology...).

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Many of the results we present in these chapters are new and are not yet available in print. These results were obtained in collaboration with G. Besson and S. Gallot (see [B-B-G1 to 3], [B-G]).

In Chapter IV, we introduce isoperimetric methods on compact Riemannian manifolds without boundary. The general setup described in § B, as well as the proof of J. Cheeger's lower bound for the first non-zero eigenvalue of a closed Riemannian manifold, are new. They arose from the above mentioned papers, and from brainstorming sessions with G. Besson and S. Gallot.

In Chapter V, we introduce the heat equation and then go directly to the main <u>tool</u> in this book: the isoperimetric inequality for the heat kernel. The ideas we develop here are those of [B-G]; our presentation differs however from that of [B-G] and is more in the spirit of Chapter IV.

Chapter VI is devoted to some applications of isoperimetric inequalities to Riemannian geometry. We use the ideas of [B-G], and the isoperimetric inequality obtained in [B-B-G1], to give bounds on topological invariants. The underlying method is the analytic method introduced by S. Bochner in the early 1940's, to obtain vanishing theorems. This method was improved by P. Li (1980) to give estimating theorems for Betti numbers, and later by S. Gallot (1981) to give estimating theorems in a more general framework. Both used isoperimetric estimates for Sobolev constants. In Chapter VI, we introduce a new idea (that of using Kato's inequality on heat kernels) which is due to M. Gromov, and came to life with the isoperimetric inequalities on the heat kernel given in [B-G]. It is important to read this chapter keeping in mind the compactness theorems of M. Gromov. These theorems are briefly described in the last paragraph of Chapter VI (see [SI] for a review).

These chapters are completed by an Appendix written by G. Besson.

In Appendix A, G. Besson shows how one can think of symmetrization procedures as relationships between Riemannian Geometry/Spectral Geometry on the one hand, and Operator Theory in a Hilbert space on the other hand; he views Kato's inequality (Chapter VI), and the symmetrization à la Faber-Krahn (Chapters IV and V), as particular cases of a unique general theorem. This interpretation is important because it distinguishes geometric techniques (isoperimetric inequalities) from analytic techniques (quadratic forms and operator theory); it also separates technicalities from fundamental ideas.

I am very grateful to G. Besson for writing this Appendix.

Spectral geometry has witnessed much research activity since the late 1960's. In Chapter VII, we very briefly sketch some of the important recent developments (in particular, see the very end of Chapter VII for more references).

Appendix B is a bibliography which I compiled in collaboration with M. Berger. I would like to thank M. Berger for allowing me to include it here. This bibliography is reproduced from the printed original; I thank the publisher Kaigai Publications (Japan) who left us the copyright. This bibliography is referred to as [B-B] in the text. It is divided into several chapters dealing with the different aspects of spectral geometry. Although the title refers to 1982, we revised the bibliography in September 1983. In Appendix C, I have added some new references.

This book was originally written both as a support for, and as a complement to lectures delivered at the 15º Colóquio Brasileiro de Matemática, July 1985. Although I have tried to give many complete proofs, I deliberately put emphasis on ideas rather than on technicalities. In a sense this book is an invitation to spectral geometry, rather than a course on spectral geometry. The original notes were first published by IMPA, in the series "Colóquio Brasileiro de Matemática". This new edition differs very little from the original one, as far as the mathematics are concerned: in order to avoid delay, I have only corrected some mistakes in the original text. In an attempt to make these notes more useful, I have added Appendix C (as a complement to [B-B]) and two indexes.

I thank the organizing Committee of the 15º Colóquio Brasileiro de Matemática for the opportunity to give a course on spectral geometry, and IMPA for its hospitality.

It is a pleasure for me to thank M.F. Cordel and P. Strazzanti who typed the first version of these notes, as well as Rogério Dias Trindade who typed the present text, for their care and competence.

I profited very much from regular brainstorming sessions with G. Besson and S. Gallot over the last three years. This book is an outgrowth of our collaboration. I owe them very much.

This book is dedicated to Marcel Berger in acknowledgement of his teachings.

Rio de Janeiro, April 1986.

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CHAPTER I

MOTIVATIONS AND THE PHYSICAL POINT OF VIEW

1. The purpose of this chapter is to introduce some basic concepts which arise naturally from problems in mathematical physics. Our presentation might appear childish...; we do not aim at establishing good mathematical models for some elasticity problems. We only want to show how the notions of energy integral, variational methods, boundary conditions, wave equation, separation of variables, eigenvalue problems... arise naturally from problems in mathematical physics, and how they are related to other fields in mathematics (partial differential equations, spectral theory, Riemannian geometry).

A. AN ELEMENTARY EXAMPLE

2. Let us consider a homogeneous elastic string S whose position at rest is represented by the line segment $[0,L]$ in the plane. The string being elastic, the tension forces are tangential to the string. The string being homogeneous, the linear density ρ and the tension μ of the string are constant along the string.

The first problem we shall deal with is that of the equilibrium position of the string S submitted to an external force which acts in the plane, transversally to the string, with intensity $f(x)$.

We represent the equilibrium position of the string by a function $u: [0,L] \to \mathbb{R}$, the amplitude of the deflection of the string, therefore assuming that the points of the string can only move transversally.

The potential energy of the string consists of two terms: the energy $E_t(u)$ which arises from the tension μ, and the external energy $E_e(u)$ which arises from the force applied to the string. The energy E_t equals the tension times the increase of length of the string; the external energy is the work of the force f. We have

$$(3)\qquad \begin{cases} E_t(u) = \mu[\displaystyle\int_0^L (1+(u_x')^2)^{1/2}dx - L]; \\ E_e(u) = \displaystyle\int_0^L f(x)u(x)dx. \end{cases}$$

We shall now make the assumption that the deflection of the string is "very small" in the sense that we can replace $(1+(u_x')^2)^{1/2}$ by $\frac{1}{2}(u_x')^2$. The potential energy of the string can then be replaced by

$$(4)\qquad E(u) = \frac{\mu}{2}\int_0^L (u_x')^2dx + \int_0^L f(x)u(x)dx.$$

In order to find u, we apply the <u>principle of least potential energy</u> which says that a stable equilibrium u is a local minimum of the energy E, which implies that

$$(5)\qquad \frac{d}{d\varepsilon}E(u+\varepsilon v)\Big|_{\varepsilon=0} = 0,$$

where $u + \varepsilon v$ represents a position of the string close to the equilibrium u.

If we plug condition (5) into (4), we find

$$(6)\qquad \mu\int_0^L u_x'v_x'dx + \int_0^L f(x)v(x)dx = 0.$$

We can of course take local variations v, i.e. variations with compact support in $]0,L[$. Taking such a variation and integrating by parts, we find that for all v in $C_o^\infty(]0,L[)$,

$$\int_0^L [-\mu u''_{xx} + f(x)]v(x)dx = 0 \quad \text{and hence}$$

(7) $$\mu \frac{d^2u}{dx^2}(x) = f(x) \quad \text{in} \quad]0,L[.$$

8. Remark. We have implicitely made the assumption that u is twice differentiable, in order to be able to write (7). We shall show how one can make weaker assumptions later on (nº 43).

9. Let us now take the function v in $C^\infty([0,L])$. Equation (6) becomes, after integration by parts,

$$\mu u'_x v\Big|_0^L + \int_0^L (f(x)-\mu u''_{xx})\, v(x)dx = 0.$$

Taking (7) into account, we then have

(10) $$u'_x(L)v(L) - u'_x(0)v(0) = 0.$$

The fact that one can take one v or another depends on the physical problem at hand. If we do not impose any condition on v, we deduce from (10) that u must satisfy the natural boundary condition (Neumann boundary condition)

(10N) $$u'_x(0) = 0 \quad \text{and} \quad u'_x(L) = 0.$$

If we assume that the string is fixed at both ends (think of a violin or a piano string), we must impose that the deflection of the string is 0 at $x = 0$ and $x = L$. This means that both u and v must satisfy the boundary condition (Dirichlet boundary condition)

(10D) $u(0) = 0$ and $u(L) = 0$.

In that case, (10) is void. The boundary condition (10N) corresponds to a free string, for which all deflections are allowed or admissible. The boundary condition (10D) corresponds to a string which is fixed at both ends. We then impose that the deflections satisfy $u(0) = 0$ and $u(L) = 0$. It is physically very intuitive that such conditions must be imposed to determine the equilibrium position of the problem under consideration.

11. Summary. In order to determine the equilibrium of a string submitted to a transversal external force f, we can

(i) either seek the local extrema of the energy

$$E(u) = \frac{\mu}{2}\int_0^L (u'_x)^2 dx + \int_0^L f(x)u(x)dx,$$

when u varies in a space of admissible functions, corresponding to the physical problem under consideration;

(ii) or solve the equation

$$\mu \frac{d^2u}{dx^2}(x) = f(x) \quad \text{in} \quad]0,L[,$$

where some boundary conditions are imposed to u at $x = 0$ and $x = L$, depending on the problem which is considered.

Examples:

Dirichlet problem (string fixed at both ends):

. Admissible functions: $u \in C^2([0,L])$ (see nº 8) such that $u(0) = u(L) = 0$ ($u+\epsilon v$ must also be admissible),

. Boundary conditions: $u(0) = 0$ and $u(L) = 0$;

Neumann problem (free string)

- Admissible functions: $u \in C^2([0,L])$ (see nº 8),
- Boundary conditions: $u'(0) = 0$ and $u'(L) = 0$ (imposed by the least potential energy principle).

12. Let us now consider the problem of the vibrating string, i.e. let us determine the laws of motion of an elastic string. We denote by $u: \mathbb{R} \times [0,L] \to \mathbb{R}$ the deflection of the string which is assumed to be transverse and small (in the sense used to derive (4)). The function f considered above may also depend on the time parameter t. We then have to consider the kinetic energy of the string, namely

$$(13) \qquad E_k(u) = \int_0^L \frac{1}{2}\rho(u_t')^2(t,x)dx.$$

Let t_1 and t_2 be two instants of time. Hamilton's principle states that the motion $u(t,x)$ of the string between the instants of time t_1 and t_2 should minimize the expression

$$J(u) = \int_{t_1}^{t_2}\int_0^L \{\frac{1}{2}\rho(\frac{\partial u}{\partial t}(t,x))^2 - \frac{1}{2}\mu(\frac{\partial u}{\partial x})^2(t,x) - f(t,x)u(t,x)\}dtdx,$$

among all admissible motions close to u, taking the same values as u at $t = t_1$, and $t = t_2$ i.e.

$$(14) \qquad \frac{d}{d\varepsilon}J(u+\varepsilon v)\Big|_{\varepsilon=0} = 0,$$

for all admissible functions v such that $v(t_1,x) = 0$ and $v(t_2,x) = 0$, for all x in $[0,L]$.

The adjective admissible refers to functions describing the physical problem under consideration as above (see nº 9 to 11).

Applying Hamilton's principle with $v \in C^\infty(\mathbb{R} \times [0,L])$ satisfying $v(t_1,x) = 0$, $v(t_2,x) = 0$, for all x, and integrating by parts, we deduce from (14) that

$$\int_{t_1}^{t_2}\int_0^L \{\rho \frac{\partial^2 u}{\partial t^2}(t,x) - \mu \frac{\partial^2 u}{\partial x^2}(t,x) + f(t,x)\}\ v(t,x)dt\ dx$$

$$+ \int_{t_1}^{t_2} \mu \frac{\partial u}{\partial x}(t,x)v(t,x)dt \Big|_0^L = 0.$$

The choice of v being arbitrary we conclude that

(15) $$\rho \frac{\partial^2 u}{\partial t^2}(t,x) - \mu \frac{\partial^2 u}{\partial x^2}(t,x) + f(t,x) = 0 \quad \text{in} \quad \mathbb{R}\times]0,L[,$$

(16) $$\frac{\partial u}{\partial x}(t,x)v(t,x)\Big|_0^L = 0 \quad \text{for all admissible} \quad v, \quad \text{and all} \quad t.$$

In the case of a string with free ends (i.e. no condition on u and v), Equation (16) gives (Neumann conditions)

(16N) $$\frac{\partial u}{\partial x}(t,0) = 0 \quad \text{and} \quad \frac{\partial u}{\partial x}(t,L) = 0 \quad \text{for all} \quad t.$$

In the case of a string with fixed ends, we must impose $u(t,0) = u(t,L) = 0$ and $v(t,0) = v(t,L) = 0$ for all t. Equation (16) is then always satisfied, and we only write the condition that u is admissible (Dirichlet conditions)

(16D) $$u(t,0) = 0 \quad \text{and} \quad u(t,L) = 0 \quad \text{for all} \quad t.$$

Equation (15) is called the one-dimensional wave equation (the space variable x being one-dimensional).

17. Remark. In order to be able to determine the motion $u(t,x)$ of the string, we need Equation (15), boundary conditions e.g. (16D) or (16N) and initial conditions; these initial conditions already appeared in the statement of Hamilton's principle; we also consider the Cauchy data $u(t_o,x) = u_o(x)$ and $u'_t(t_o,x) = u_1(x)$, $0 \le x \le L$,

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