Algebraic Geometry

ALGEBRAIC GEOMETR

Proceedings of the Conference at Berlin
9–15 March 1988

Edited by

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Preface

The Conference on Algebraic Geometry, held in Berlin 9–15 March 1988, was organised by the Sektion Mathematik of the Humboldt-Universität. The organising committee consisted of H. Kurke, W. Kleinert, G. Pfister and M. Roczen. The Conference is one in a series organised by the Humboldt-Universität at regular intervals of two or three years, with the purpose of providing a meeting place for mathematicians from eastern and western countries.

The present volume contains elaborations of part of the lectures presented at the Conference and some articles on related subjects. All papers were subject to the regular refereeing procedure of *Compositio Mathematica*, and H. Kurke acted as a guest editor of this journal.

The papers focus on actual themes in algebraic geometry and singularity theory, such as vector bundles, arithmetical algebraic geometry, intersection theory, moduli and Hodge theory.

We are grateful to all those who, by their hospitality, their presence at the Conference, their support or their written contributions, have made this Conference to a success.

The editors

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2-Cocycles and Azumaya algebras under birational transformations of algebraic schemes

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The basic question whether the injection $\mathrm{Br}(X) \to H^2(X, \mathcal{O}_X^*)_{\mathrm{tors}}$ is an isomorphism arose at the very definition of the Brauer group of an algebraic scheme X. Positive answers are known in the following cases:

1. the topological Brauer group $\mathrm{Br}(X_{\mathrm{top}}) \cong H^2(X, \mathcal{O}_{\mathrm{top}}^*)_{\mathrm{tors}} \cong H^3(X, \mathbb{Z})_{\mathrm{tors}}$ (J.-P. Serre); in the etale (algebraic) case the isomorphism is proved for
2. smooth projective surfaces (A. Grothendieck);
3. abelian varieties;
4. the union of two affine schemes (R. Hoobler, O. Gabber).

The first author has formulated a birational variant of the basic question, while considering the unramified Brauer group in [1]. The group $\mathrm{Br}_V(K(X)) = \cap \mathrm{Br}(A_v) \subseteq \mathrm{Br}(K(X))$ (intersection taken over all discrete valuation subrings A_v of the rational function field $K(X)$) is isomorphic to $H^2(\tilde{X}, \mathcal{O}^*)$, where $\tilde{X}$ is a nonsingular projective model of X, i.e. a nonsingular projective variety birationally equivalent to X.

QUESTION. Given a cocycle class $\gamma \in H^2(\tilde{X}, \mathcal{O}^*)$, is it possible to find a nonsingular projective model $\bar{X}$ such that γ is represented by a $\mathbb{P}^n$-bundle (i.e. by an Azumaya algebra) on $\bar{X}$?

The case where X is a nonsingular projective model of V/G, with G a γ-minimal group and V a faithful representation of G, was considered in [2]. O. Gabber in his letter to Bogomolov (12.1.1988) has given an affirmative answer to the question in the case of general algebraic spaces. In this paper we give a simple version of his proof for algebraic schemes.

Let X be a scheme, $\gamma \in H^2(X, \mathcal{O}^*)$, $\{U_i\}$ an affine cover of X. Then the restriction of γ to each U_i is represented by an Azumaya algebra A_i. If we would have isomorphisms $A_{i|U_i \cap U_j} \cong A_{j|U_i \cap U_j}$, we could glue the sheaves $\{A_j\}$ and get an

Azumaya algebra on X, representing γ. But we have isomorphisms $A_{i|U_i \cap U_j} \otimes \mathrm{End}(E_{ij}) \cong A_{j|U_i \cap U_j} \otimes \mathrm{End}(E_{ji})$ for certain vector bundles E_{ij}, E_{ji} on $U_i \cap U_j$.

THEOREM. *Let X be a noetherian scheme, $\gamma \in H^2(X, \mathcal{O}_X^*)$. There exists a proper birational morphism $\alpha: \tilde{X} \to X$ such that $\alpha^*(\gamma)$ is represented by an Azumaya algebra on $\tilde{X}$.*

Proof. It is enough to consider X which are connected. Suppose that $\{U_i\}$ is an affine open cover of X and that γ is non-trivial on at least one U_i. We will construct an Azumaya algebra on a birational model of X by an inductive process which involves adjoining one by one proper preimages of the subsets U_i and, by an appropriate birational change of the scheme and Azumaya algebra obtained, extending the new algebra to the union. We start with some affine open subset U_0 and an Azumaya algebra A_0 on it.

Now suppose by induction that we already have an Azumaya algebra $\bar{A}_k$ on the scheme X_k, a Zariski-open subset of the scheme $\bar{X}_k$, equipped with a proper birational map $\bar{\alpha}_k: \bar{X}_k \to X$ such that $X_k = \bar{\alpha}_k^{-1}(U_0 \cup \cdots \cup U_k)$. Let U_{k+1} intersect $U_0 \cup \cdots \cup U_k$ and $\bar{U}_{k+1} = \bar{\alpha}^{-1}(U_{k+1})$. Suppose that on U_{k+1}, γ is represented by the Azumaya algebra A_{k+1}. In the same vein as above we have an isomorphism

$$\bar{A}_{k|X_k \cap \bar{U}_{k+1}} \otimes \mathrm{End}(E_{k,k+1}) \cong \bar{\alpha}_k^*(A_{k+1})|_{X_k \cap \bar{U}_{k+1}} \otimes \mathrm{End}(E_{k+1,k})$$

and we need to extend $E_{k,k+1}$ to X_k and $E_{k+1,k}$ to $\bar{U}_{k+1}$ from their intersection. After this we will change $\bar{A}_k$ and $\bar{\alpha}_k^*(A_{k+1})$ by the other representatives $\bar{A}_k \otimes \mathrm{End}(E_{k,k+1})$, $\bar{\alpha}_k^*(A_{k+1}) \otimes \mathrm{End}(E_{k+1,k})$ of the same Brauer classes and glue these Azumaya algebras, hence the proof.

First, extend both sheaves E as coherent sheaves. This can be done by the following

LEMMA. *Let X be a noetherian scheme, $U \subseteq X$ a Zariski-open subset, E a coherent sheaf on U. Then there exists a coherent sheaf E' on X such that $E'|_U \cong E$.* This is Ex. II.5.15 in [4].

Note that we can assume that in our inductive process we add neighborhoods U_{k+1} of no more than one irreducible component (or an intersection of irreducible components) of X, different from those contained in X_k. Thus we assume $X_k \cap U_{k+1}$ to be connected and the rank of E to be constant on $X_k \cap U_{k+1}$, hence E' will be locally generated by n elements, where n is the rank of E.

LEMMA (see [3], Lemma 3.5). *Let X be a noetherian scheme, E a coherent sheaf on X, locally free outside a Zariski closed subset Z on X. Then there exists a coherent sheaf I of ideals on X such that the support of $\mathcal{O}_X/I$ is Z with the following* *property. Let $\alpha: \tilde{X} \to X$ be the blowing up of X with center I, then the sheaf $\tilde{\alpha}(E) :=$ the quotient of $\alpha^*(E)$ by the subsheaf of sections with support in $\alpha^{-1}(Z)$, is locally free on $\tilde{X}$.*

Proof. The proof consists of two parts. First: to reduce the number of local generators to get this number constant on the connected components of X (the minima are the values of the (local) rank function of E). Second, to force the kernel of the (local) presentations $\mathcal{O}_V^m \to E|_V \to 0$ to vanish for all neighborhoods from some cover $\{V\}$. Both parts are proved by indicating the suitable coherent sheaves of ideals and blowing up X with respect to these sheaves. Let $\mathcal{O}_V^m \xrightarrow{f} E|_V - 0$ be a local presentation of E. Then $\mathrm{Ker}(f)$ is generated by all relations $\sum_{i=1}^m c_i a_i = 0$ where $\{a_i\}$ stand for the free basis of $\mathcal{O}_V^m$. The coherent sheaf of ideals in the first case is the sheaf defined locally as the ideal I_V in $\mathcal{O}_V$ generated by all c_i such that $\sum_{i=1}^m c_i a_i \in \mathrm{Ker}(f)$ and in the second case as $J_X = \mathrm{Ann}(\mathrm{Ker}(f))$. As the number of generators is constant in the case we are interested in, we give the details only for the second part of the proof and refer to [3] for the first.

Let $\alpha: X' \to X$ be the blowing up of X with respect to J_X and let $\tilde{\alpha}(E)$ be as in the statement of the Lemma. Let

$$0 \to (\mathrm{Ker}(\bar{f}))|_{V'} \to \mathcal{O}_{V'}^m \xrightarrow{\bar{f}} \tilde{\alpha}(E)|_{V'} \to 0$$

be the local presentation of $\tilde{\alpha}(E)$. We have $\alpha^{-1}(\mathrm{Ann}(f)) \subseteq \mathrm{Ann}(\mathrm{Ker}(\bar{f}))$. Let $\mathfrak{p} \in Z'$, $V' = \mathrm{Spec}(A')$ an affine neighborhood of $\mathfrak{p}$ in X' and let $\sum_{i=1}^m c_i a_i \in \mathrm{Ker}(\bar{f})|_{V'}$ map to a nonzero element in $\mathrm{Ker}(\bar{f})_{\mathfrak{p}}$. Denote by γ a generator of the invertible sheaf $\alpha^{-1}(\mathrm{Ann}(f))$ on $V'' = \mathrm{Spec}(A'') \subseteq V'$ for suitable A''. It is clear that there exists for given $\mathfrak{p}$ and V'' a finite sequence of open affine neighborhoods $V_1'', \ldots, V_s''$ such that $X' \backslash Z' = V_1''$, $V'' = V_s''$ and $V_j'' \cap V_{j+1}'' \neq \emptyset$ for $j = 1, \ldots, s-1$. So suppose $V' \cap (X' \backslash Z') \neq \emptyset$ and $\mathfrak{q} \in V'' \cap (X \backslash Z)$. Then $(c_i)_{\mathfrak{q}} = 0$ for $i = 1, \ldots, m$ and $\mathfrak{q} \in \mathrm{Spec}(A_\gamma'')$ hence $\gamma^k c_i = 0$ for $i = 1, \ldots, m$ for some k. Since γ is not a zero divisor, we conclude that $c_i = 0$ for $i = 1, \ldots, m$. Thus (maybe after considering a finite sequence of points $\mathfrak{q}_1, \ldots, \mathfrak{q}_s$) we prove that $(\mathrm{Ker}(f))_{\mathfrak{p}}$ is trivial for every $\mathfrak{p} \in X'$. □

In this way we glue the two sheaves $\bar{A}_k$ and A_{k+1} and get an Azumaya algebra on $\tilde{X}_k \cup \tilde{U}_{k+1}$. As the scheme X is quasi-compact, we obtain an Azumaya algebra on $\tilde{X}$ after a finite number of such steps.

Now we have to show that this process can be done in such a way that the class [A] of the Azumaya algebra A constructed in this way is equal to $\tilde{\alpha}^*(\gamma)$. Again this goes by induction on k. We have $X_{k+1} = U \cup V$ with $U = \tilde{\alpha}_{k+1}^{-1}(U_0 \cup \cdots \cup U_k)$ and $V = \tilde{\alpha}_{k+1}^{-1}(U_{k+1})$. We have the exact sequence

$$H^1(U \cap V, \mathcal{O}^*) \to H^2(X_{k+1}, \mathcal{O}^*) \to H^2(U, \mathcal{O}^*) \oplus H^2(V, \mathcal{O}^*)$$

and by induction hypothesis, $\bar{\alpha}^*_{k+1}(\gamma) - [\bar{A}_{k+1}]$ maps to zero in $H^2(U, \mathcal{O}^*) \oplus H^2(V, \mathcal{O}^*)$ so it comes from $\beta \in H^1(U \cap V, \mathcal{O}^*)$. By blowing up X_{k+1} we may assume that β is represented by a line bundle which extends to U. Then β maps to zero in $H^2(X_{k+1}, \mathcal{O}^*)$, hence $\bar{\alpha}^*_{k+1}(\gamma) - [\bar{A}_{k+1}] = 0$. □

Note that we need not bother about the compatibility of isomorphisms, because at each step we choose a new isomorphism between the Azumaya algebra A on $U_1 \cup \cdots \cup U_j$ from the preceding step and A_k on U_k, modulo End(E), End(E_k).

COROLLARY 1. *Let G be a finite group, V a faithful complex representation of G. Then there exists a nonsingular projective model X of V/G such that* $\mathrm{Br}(X) = H^2(X, \mathcal{O}^*)$.

Proof. The group $H^2(X, \mathcal{O}^*)$ is a birational invariant of nonsingular projective varieties and is isomorphic to $H^2(G, \mathbb{Q}/\mathbb{Z})$ if X is a model of V/G (see [1]). It remains to recall that the group $H^2(G, \mathbb{Q}/\mathbb{Z})$ is finite. □

COROLLARY 2. *Let X be a noetherian scheme over $\mathbb{C}$, Z a closed subscheme of X and $\gamma \in H^2_Z(X, \mathcal{O}^*)$. Then there exists a proper morphism $\alpha: X' \to X$ which is an isomorphism above $X \backslash Z$ and maps γ to zero in $H^2_{\alpha^{-1}(Z)}(X', \mathcal{O}^*)$.*

Proof. First, let's have $\alpha(\gamma)$ map to zero in $H^2(X, \mathcal{O}^*)$. To do this, desingularize X by $X' \to X$. Then in the following exact sequence (in etale cohomology), β will be injective:

$$\begin{array}{ccccccc} H^1(X'\backslash Z', \mathcal{O}^*) & \to & H^2_{Z'}(X', \mathcal{O}^*) & \to & H^2(X', \mathcal{O}^*) & \xrightarrow{\beta} & H^2(X'\backslash Z', \mathcal{O}^*) \\ & & & & \uparrow & & \\ & & & & H^2_Z(X, \mathcal{O}^*) & & \end{array}$$

The injectivity is due to the injectivity of $H^2(X', \mathcal{O}^*) \to H^2(K(X'), \mathcal{O}^*)$ for a nonsingular irreducible scheme X'.

Now γ comes from $\gamma' \in H^1(X'\backslash Z', \mathcal{O}^*) = \mathrm{Pic}(X'\backslash Z')$. It is obvious that Picard elements lift to Picard elements by the blowing ups from the theorem. Thus from the diagram

$$\begin{array}{ccccc} H^1(X'', \mathcal{O}^*) & \to & H^1(X''\backslash Z'', \mathcal{O}^*) & \to & H^2_{Z''}(X'', \mathcal{O}^*) \\ & & \uparrow & & \uparrow \\ & & H^1(X'\backslash Z', \mathcal{O}^*) & \to & H^2_{Z'}(X', \mathcal{O}^*) \end{array}$$

we conclude that γ becomes trivial on Z'' by $X'' \to X'$ which extends γ' to X''. □

Now let us return to the problem of an isomorphism $\mathrm{Br}(X) \to H^2(X, \mathcal{O}^*)$ for nonsingular quasi-projective varieties. The theorem reduces the general problem to the following

QUESTION. Let X' be a blowing up of a nonsingular variety X along a smooth subvariety S and let A' be an Azumaya algebra on X'. Does there exist an Azumaya algebra A on X such that its inverse image on X' is equivalent to A'?

In case the restriction of A' to the pre-image of S is trivial, the question reduces to the one, whether a vector bundle on this preimage can be extended to X as a vector bundle. For example, if $\dim(X) = 2$ then S is a point and its proper preimage is a $\mathbb{P}^1$ with self-intersection -1. Since the map $\mathrm{Pic}(X') \to \mathrm{Pic}(\mathbb{P}^1)$ is surjective, any vector bundle on $\mathbb{P}^1$ can be extended to X'.

Therefore we obtain a simple proof of the basic theorem in the case $\dim(X) = 2$ using the birational theorem.

In the case of $\dim(X) = 3$ the same procedure reduces the basic problem to the analogous problem of extending vector bundles from $\mathbb{P}^2$ and ruled surfaces to a variety of dimension three.

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Monads and cohomology modules of rank 2 vector bundles

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Introduction

Monads are a useful tool to construct and study rank 2 vector bundles on the complex projective space $\mathbb{P}_n$, $n \geqslant 2$ (compare [O-S-S]). Horrocks' technique of eliminating cohomology [Ho 2] represents a given rank 2 vector bundle $\mathscr{E}$ as the cohomology of a monad

$$(M(\mathscr{E})) \quad \mathscr{A} \xrightarrow{\varphi} \mathscr{B} \xrightarrow{\psi} \mathscr{C}$$

as follows.

First eliminate the graded $S = \mathbb{C}[z_0, \ldots, z_n]$-module $H^1\mathscr{E}(*) = \bigoplus_{m \in \mathbb{Z}} H^1(\mathbb{P}_n, \mathscr{E}(m))$ by the universal extension

$$0 \to \mathscr{E} \to \mathscr{Q} \to \tilde{L}_0 \to 0,$$

where

$$L_0 \to H^1\mathscr{E}(*) \to 0.$$

is given by a minimal system of generators (~ stands for sheafification).

If $n = 2$ take this extension as a monad with $\mathscr{A} = 0$.

If $n \geqslant 3$ eliminate dually $H^{n-1}\mathscr{E}(*)$ by the universal extension

$$0 \to \tilde{L}_0^{\vee}(c_1) \to \mathscr{K} \to \mathscr{E} \to 0$$

(where $c_1 = c_1(\mathscr{E})$ is the first Chern-class). Then notice, that the two extensions

* Partially supported by the DAAD.

can be completed to the display

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \tilde{L}_0^{\vee}(c_1) & \longrightarrow & \mathscr{K} & \longrightarrow & \mathscr{E} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \tilde{L}_0^{\vee}(c_1) & \xrightarrow{\varphi} & \mathscr{B} & \longrightarrow & \mathscr{Q} & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle\psi} & & \downarrow & & \\
 & & & & \tilde{L}_0 & = & \tilde{L}_0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

of a monad

$$\tilde{L}_0^{\vee}(c_1) \xrightarrow{\varphi} \mathscr{B} \xrightarrow{\psi} \tilde{L}_0$$

for $\mathscr{E}$.

To get a better understanding for $\mathscr{B}$, φ and ψ consider first the case $n = 2, 3$. Then $\mathscr{B}$ is a direct sum of line bundles by Horrocks' splitting criterion [Ho 1]. Taking cohomology we obtain a free presentation

$$B \xrightarrow{\psi} L_0 \longrightarrow H^1\mathscr{E}(*) \longrightarrow 0$$

with $B = H^0\mathscr{B}(*)$. The crucial point is that this is minimal [Ra]. Moreover, if $n = 3$, then B is self-dual [Ra]: $B^{\vee}(c_1) \simeq B$. We will see below that up to isomorphism φ is the dual map of ψ.

Let us summarize and slightly generalize. Consider an arbitrary graded S-module N of finite length with minimal free resolution (m.f.r. for short)

$$0 \longrightarrow L_{n+1} \xrightarrow{\alpha_n} L_n \longrightarrow \cdots \longrightarrow L_1 \xrightarrow{\alpha_0} L_0 \longrightarrow N \longrightarrow 0.$$

If $n = 2$ then $N \simeq H^1\mathscr{E}(*)$ for some rank 2 vector bundle $\mathscr{E}$ on $\mathbb{P}_2$ iff $\operatorname{rk} L_1 = \operatorname{rk} L_0 + 2$ (compare [Ra]). In this case $\mathscr{E}$ is uniquely determined as $\widetilde{\ker \alpha_0}$:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \tilde{L}_0^{\vee}(c_1) & \xrightarrow{\alpha_0^{\vee}(c_1)} & \tilde{L}_1^{\vee}(c_1) & & \xrightarrow{\alpha_1} & \tilde{L}_1 & \xrightarrow{\alpha_0} & \tilde{L}_0 & \longrightarrow 0. \\
 & & & & & \searrow & \nearrow & & & & \\
 & & & & & & \mathscr{E} & & & & \\
 & & & & & \nearrow & \searrow & & & & \\
 & & & & 0 & & & 0 & & &
\end{array}$$

(This sequence is self-dual by Serre-duality [Ho 1, 5.2], since $\mathscr{E}^\vee(c_1) \simeq \mathscr{E}$).

For $n = 3$ there is an analogous result. Answering Problem 10 of Hartshorne's list [Ha] we prove:

PROPOSITION 1. *N is the first cohomology module of some rank 2 vector bundle on $\mathbb{P}_3$ iff*

(1) $\operatorname{rk} L_1 = 2 \operatorname{rk} L_0 + 2$ *and*
(2) *there exists an isomorphism* $\Phi\colon L_1^\vee(c_1) \xrightarrow{\simeq} L_1$ *for some* $c_1 \in \mathbb{Z}$ *such that* $\alpha_0 \circ \Phi \circ \alpha_0^\vee(c_1) = 0$.

In this case any Φ satisfying (2) *defines a monad*

$$(M_\Phi) \quad \tilde{L}_0^\vee(c_1) \xrightarrow{\Phi\circ\alpha_0^\vee(c_1)} \tilde{L}_1 \xrightarrow{\alpha_0} \tilde{L}_0$$

and $\mathscr{E}$ is a 2-bundle on $\mathbb{P}_3$ with $H^1\mathscr{E}(*) \simeq N$ (and $c_1 = c_1(\mathscr{E})$) iff $(M(\mathscr{E})) \simeq (M_\Phi)$ for some Φ.

To complete the picture let us mention a result of Hartshorne and Rao (not yet published). If $N \simeq H^1\mathscr{E}(*)$ as above then $L_0^\vee(c_1) \xrightarrow{\varphi} L_1$ is part of a minimal system of generators for $\ker \alpha_0$. In other words: There exists a splitting

$$0 \to L_4 \to L_3 \to L_2' \oplus L_0^\vee(c_1) \to L_1 \to L_0 \to H^1\mathscr{E}(*) \to 0$$

inducing the monad

$$(M(\mathscr{E})) \quad \tilde{L}_0^\vee(c_1) \to \tilde{L}_1 \to \tilde{L}_0$$

and the m.f.r.

$$0 \to \tilde{L}_4 \to \tilde{L}_3 \to \tilde{L}_2' \to \mathscr{E} \to 0$$

resp.

For $n \geq 4$ there is essentially only one indecomposable 2-bundle known on $\mathbb{P}_n$: The Horrocks-Mumford-bundle $\mathscr{F}$ on $\mathbb{P}_4$ with Chern-classes $c_1 = -1$, $c_2 = 4$. We prove:

PROPOSITION 2. *The* m.f.r. *of $H^2\mathscr{F}(*)$ decomposes as*

$$0 \to H_2 \xrightarrow{\beta_1} H_1 \xrightarrow{\begin{pmatrix}\beta_0' \\ \beta_0''\end{pmatrix}} L_0^\vee(c_1) \oplus L_1 \xrightarrow{\begin{pmatrix}0 & \alpha_0 \\ \alpha_0^\vee(c_1) & *\end{pmatrix}} L_0 \oplus L_1^\vee(c_1) \to$$

$$L_0^\vee(c_1) \oplus L_1 \searrow B \nearrow L_0 \oplus L_1^\vee(c_1), \qquad 0 \nearrow B \searrow 0$$

$$\to H_1^\vee(c_1) \to H_2^\vee(c_1) \to H^2\mathscr{F}(*) \to 0$$

with $B = H^0\mathscr{B}(*)$, *inducing the monad*

$$(M(\mathscr{F})) \quad \tilde{L}_0^{\vee}(c_1) \to \mathscr{B} \to \tilde{L}_0$$

and the minimal free presentation

$$L_1 \xrightarrow{\alpha_0} L_0 \longrightarrow H^1\mathscr{F}(*) \longrightarrow 0.$$

The corresponding m.f.r. *decomposes as*

$$0 \to L_5 \to L_4 \to L_3' \oplus H_2 \xrightarrow{\begin{pmatrix} * & 0 \\ * & \beta_1 \end{pmatrix}} L_2' \oplus H_1 \xrightarrow{(* \ \ \beta_0'')} L_1 \xrightarrow{\alpha_0} L_0 \to$$
$$\to H^1\mathscr{F}(*) \to 0$$

inducing the m.f.r.

$$0 \to \tilde{L}_5 \to \tilde{L}_4 \to \tilde{L}_3' \to \tilde{L}_2' \to \mathscr{F} \to 0.$$

$(M(\mathscr{F}))$ is the monad given in [H-M]. Using its display we can compute the above m.f.r.'s explicitly. Especially we reobtain the equations of the abelian surfaces in $\mathbb{P}_4$ ([Ma 1], [Ma 2]).

Of course we may deduce from $\mathscr{F}$ some more bundles by pulling it back under finite morphisms $\pi: \mathbb{P}_4 \to \mathbb{P}_4$. The above result also holds for the bundles $\pi^*\mathscr{F}$ with $(M(\pi^*\mathscr{F})) = \pi^*(M(\mathscr{F}))$.

There is some evidence (but so far no complete proof), that a splitting as in Proposition 2 occurs for every indecomposable 2-bundle on $\mathbb{P}_4$. This suggests a new construction principle for such bundles by constructing their H^2-module first.

Proof of Proposition 1

Let $n = 3$ and N be a graded S-module of finite length with m.f.r.

$$0 \to L_4 \to L_3 \to L_2 \to L_1 \xrightarrow{\alpha_0} L_0 \to N \to 0.$$

Suppose first that $N \simeq H^1\mathscr{E}(*)$ for some 2-bundle $\mathscr{E}$ on $\mathbb{P}_3$ (with first Chern-class c_1). As seen in the introduction, Horrocks' construction leads to a monad

$$(M(\mathscr{E})) \quad \tilde{L}_0^{\vee}(c_1) \xrightarrow{\varphi} \tilde{L}_1 \xrightarrow{\alpha_0} \tilde{L}_0$$

for $\mathscr{E}$. The dual sequence

$$\tilde{L}_0^\vee(c_1) \xrightarrow{\alpha_0^\vee(c_1)} \tilde{L}_1^\vee(c_1) \xrightarrow{\varphi^\vee(c_1)} \tilde{L}_0$$

is a monad for $\mathscr{E}^\vee(c_1) \simeq \mathscr{E}$. The induced presentation of N has to be isomorphic to that one given by the m.f.r.:

$$\begin{array}{ccccccc} L_1^\vee(c_1) & \xrightarrow{\varphi^\vee(c_1)} & L_0 & \to & N & \to & 0 \\ {\scriptstyle \Phi^\vee(c_1)}\downarrow \;\; \wr\| & & \wr\| & & \wr\| & & \\ L_1 & \xrightarrow{\alpha_0} & L_0 & \to & N & \to & 0. \end{array}$$

Dualizing gives (2) since $\alpha_0 \circ \varphi = 0$ and thus also a monad (M_Φ) for $\mathscr{E}$, isomorphic to $(M(\mathscr{E}))$ (replace φ by $\Phi \circ \alpha_0^\vee(c_1)$).

Conversely if N satisfies (2), we obtain a monad (M_Φ) by sheafification. (Since $\tilde{N} = 0$, α_0 is a bundle epimorphism. Dually $\alpha_0^\vee(c_1)$ is a bundle monomorphism.) Let $\mathscr{E}$ be the cohomology bundle of (M_Φ). Then $H^1\mathscr{E}(*) \simeq N$. $\mathscr{E}$ has rank 2, if N satisfies (1). □

REMARK 1. (i) Let $N \simeq H^1\mathscr{E}(*)$ as above with induced splitting

$$0 \to L_4 \to L_3 \to L_2' \oplus L_0^\vee(c_1) \to L_1 \to L_0 \to H^1\mathscr{E}(*) \to 0$$

as in the introduction. Recall that $\mathscr{E}$ is stable iff $H^0(\mathbb{P}_3, \mathscr{E}(m)) = 0$ for $m \leqslant -c_1/2$. Thus $\mathscr{E}$ is stable iff L_2' has no direct summand $S(m)$ with $m \geqslant c_1/2$. Notice that this condition only depends on N.

(ii) If N satisfies (1) and has only one generator, then (2) is obviously equivalent to the symmetry condition $L_1^\vee(c_1) \simeq L_1$. Thus [Ra, 3.1] is a special case of Proposition 1.

EXAMPLES. (i) The well-known Null correlation bundles are by definition the bundles corresponding to the S-module $\mathbb{C}$. Consider the Koszul-presentation

$$4S \xrightarrow{\alpha_0} S(1) \to \mathbb{C} \to 0, \quad \alpha_0 = (z_0, z_1, z_2, z_3).$$

The isomorphisms $4S \xrightarrow{\Phi} 4S$ with $\alpha_0 \circ \Phi \circ \alpha_0^\vee(c_1) = 0$ are precisely the 4×4 skew symmetric matrices with nonzero determinant. Two such matrices give isomorphic bundles iff they differ by a scalar (use [O-S-S, II, Corollary 1 to 4.1.3]). The moduli space of Null correlation bundles is thus isomorphic to $\mathbb{P}_5 \backslash \mathbb{G}$, where $\mathbb{G}$ is the Plucker embedded Grassmanian of lines in $\mathbb{P}_3$.

Unlike the case $n = 2$ the bundle is not uniquely determined by the module.