Number Theory and Physics

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Preface

Les Houches

Number theory, or arithmetic, sometimes referred to as the queen of mathematics, is often considered as the purest branch of mathematics. It also has the false reputation of being without any application to other areas of knowledge. Nevertheless, throughout their history, physical and natural sciences have experienced numerous unexpected relationships to number theory. The book entitled *Number Theory in Science and Communication*, by M.R. Schroeder (Springer Series in Information Sciences, Vol. 7, 1984) provides plenty of examples of cross-fertilization between number theory and a large variety of scientific topics.

The most recent developments of theoretical physics have involved more and more questions related to number theory, and in an increasingly direct way. This new trend is especially visible in two broad families of physical problems. The first class, dynamical systems and quasiperiodicity, includes classical and quantum chaos, the stability of orbits in dynamical systems, K.A.M. theory, and problems with "small denominators", as well as the study of incommensurate structures, aperiodic tilings, and quasicrystals. The second class, which includes the string theory of fundamental interactions, completely integrable models, and conformally invariant two-dimensional field theories, seems to involve modular forms and p-adic numbers in a remarkable way.

Several theoretical physicists, having experienced this evolution, have found it worthwhile to bring their individual and occasional relations with mathematicians working in number theory into a more official and profitable framework. This idea has met with a very positive echo from the mathematicians' side. Thus a first version of the scientific committee given below gathered in an informal way in order to organise an interdisciplinary meeting on "Number Theory and Physics". Our aim was twofold. Physicists were given, mostly through the programme of lectures, the opportunity to become acquainted with the basic methods and results of number theory that are the most used in their branch. Mathematicians discovered in most of the seminars novel domains of the physical sciences where methods and results related to arithmetic have been useful in recent years.

The meeting was held at the Centre de Physique, Les Houches (Haute-Savoie, France), March 7 – 16, 1989. Most of the 58 participants, both mathematicians and physicists, attended the totality of the scientific programme, in spite of the charm of the mountainous setting, the fine weather and the neighbouring ski slopes.

Although the timetable was already rather dense, with 32 lectures and 33 seminars, many informal, but passionate, discussions and meetings often lasted late into the night.

The Scientific Committee that organised *Number Theory and Physics* was composed of eight French scientists: J. Bellissard (Theoretical Physics, Marseille), C. Godrèche (Solid State Physics, CEN Saclay), C. Itzykson (Theoretical Physics, CEN Saclay), J.-M. Luck (Theoretical Physics, CEN Saclay), M. Mendès France (Mathematics, Bordeaux), P. Moussa (Theoretical Physics, CEN Saclay), E. Reyssat (Mathematics, Caen), and M. Waldschmidt (Mathematics, IHP, Paris). The organisers were assisted by an International Advisory Committee, composed of M. Berry (Physics, Bristol, UK), P. Cvitanović (Physics, Copenhagen, Denmark), M. Dekking (Mathematics, Delft, The Netherlands), and G. Turchetti (Physics, Bologna, Italy).

The following institutions are most gratefully acknowledged for their generous specific financial support of the meeting: the Département Mathématiques et Physique de Base of the Centre National de la Recherche Scientifique; the Institut de Recherche Fondamentale of the Commissariat à l'Energie Atomique; the Direction des Recherches, Etudes et Techniques de la Délégation Générale pour l'Armement (under contract no. 88/1474); the French Ministère de l'Education Nationale; the French Ministère des Affaires Etrangères; and the Commission of the European Communities. The regretted absence of support from NATO finally turned out to allow a more flexible organisation.

The variety and abundance of the material presented during lectures and seminars led to the decision to publish two separate volumes of proceedings. This first volume is devoted to the proceedings of the seminars. These talks, limited on purpose to 40 minutes, offered to most participants, both mathematicians and physicists, the possibility of illustrating, through their own research activity, the variety of specific physical contexts which involve number-theoretic questions in a natural though sometimes unexpected way. The 32 contributions to this volume have been grouped into five parts for the sake of clarity: (I) conformally invariant field theories, integrability, quantum groups; (II) quasicrystals and related geometrical structures; (III) spectral problems, automata, and substitutions; (IV) dynamical and stochastic systems; (V) further arithmetical problems, and their relationship to physics.

An extensive programme of 32 lectures gave to all participants a rather complete view of the different areas of number theory that, according to us, play a role in physics. The first intensive set of 24 lectures made physicists more familiar with basic concepts and methods in the branches of number theory that have recently turned out to be useful in physics. The contributors were F. Beukers, J.B. Bost, P. Cartier, G. Christol, H. Cohen, R. Gergondey, E. Reyssat, H.M. Stark, and D. Zagier. The subjects covered included p-adic numbers, Galois theory and number fields, elliptic functions and modular forms. A second series of lectures displayed some theoretical physics problems where number theory plays a prominent role, namely the quasiperiodicity theory in dynamical systems and quasicrystals. The contributors were J. Bellissard, P. Cvitanović, M. Duneau, A. Katz, M. Senechal and J.C. Yoccoz. We plan to publish a written version of these lectures in a separate volume.

Although "Number Theory and Physics" may have seemed to be both too ambitious and too exotic a theme for a 10-day workshop, this meeting has been quite fruitful, according to the participants, in spite of its brevity, and allows one to expect an even richer variety of subjects of future collaborations between mathematicians and physicists, as well as other meetings around similar themes.

Saclay and Paris
June 1989

J.-M. Luck
P. Moussa
M. Waldschmidt

Word of Appreciation

On behalf of those who have attended this Workshop I wish to thank Jean-Marc Luck and the other organisers of this meeting. It is always a great pleasure for me to revisit Les Houches. I greatly enjoy the intense concentration on the subject which is characteristic of meetings here, combined with the opportunity to enjoy the beautiful surroundings. I have found this to be an enormously stimulating meeting. In this group of mathematicians and physicists from different disciplines we have had to face the challenge of understanding other languages and unfamiliar sets of priorities, and we have had to translate our own enthusiasms into language that people from other disciplines can understand. This has all been made easier by the smooth running of the Workshop by the organisers.

David Thouless

Contents

The talks by Professor M. Berry and Professor P. Freund are not reported in this volume because most of the material presented has already appeared in the following:

M.V. Berry, J. Goldberg: Renormalisation of curlicues. Nonlinearity *1*, 1–26 (1988)

M.V. Berry: Semiclassical formula for the number variance of the Riemann zeros. Nonlinearity *1*, 399–407 (1988)

P.G.O. Freund: "The Arithmetic of Strings", in *New Theories in Physics*, ed. by Z. Ajduk, S. Pokorski, A. Trautman (World Scientific, Singapore 1989) pp. 369–382

Part I

Conformally Invariant Field Theories, Integrability, Quantum Groups

Z/NZ Conformal Field Theories

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Abstract : In this paper, we give a brief introduction to Conformal Field Theory (CFT) following the presentation of G. Segal. We explain how to reconstruct part of a CFT from its fusion rules. The possible choices of S matrices are indexed by some automorphisms of the fusion algebra. We illustrate this procedure by computing the modular properties of the possible genus-one characters when the fusion algebra is the representation algebra of a finite group. We also classify the modular invariant partition functions of these theories. We recover as special cases the $A_N^{(1)}$ WZW theories and the rational gaussian model.

Introduction

It is now clear that the last few years have witnessed a dramatic convergence of interests and methods between mathematics and theoretical physics. String theory and more recently, Conformal Field Theory perfectly illustrate this evolution.

CFT is believed to be of basic interest both in particle physics and statistical mechanics. According to Friedan [1], the classical states of string theory (i.e. the propagation of a string in a fixed background space-time) are in one-to-one correspondence with Conformal Field Theories in two dimensions; and any statistical system with a second order phase transition is scale, translationally and rotationally invariant at its critical point which is therefore described by a Conformal Field Theory.

From a mathematical point of view, the importance of CFT in two dimensions arises from the enormous symmetry of these theories. It is so huge that the hope of a possible classification has emerged. The main idea is to classify the states of a CFT using representations of the symmetry algebra of the theory. This has pointed out a connection between CFT and the representation theory of infinite dimensional algebras among which are the Virasoro algebra and the Kac-Moody algebras and their supersymmetric generalisations. Different formulations of a CFT led to various identities between partition functions and connected this field to number theory [1]. More recent works concerning the study of CFT's on arbitrary Riemann surfaces relate CFT to the study of Riemann surfaces and modular geometry [2][3][4][5]. As was shown by J. Birman [6], the modular group of an arbitrary Riemann surface is related to a braid group and CFT's naturally provide us with representations of the braid group, quantum groups and integrable models and, following E.Witten [7], of knot theory and topological field theory.

[1] The reader will remember the celebrated Dyson-Mac Donald identities

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This paper is an attempt to give a very brief introduction to Conformal Field Theory for mathematicians. We do not pretend to be either exhaustive or rigorous but rather to indicate the ideas and methods used in this field. In section 1, we review some basic aspects of conformal field theory. We follow the presentation of Segal [8]: a CFT associates a state of a certain Hilbert space with each Riemann surface with boundaries. We describe how this space arises from the representation theory of the chiral algebra. The case of Rational Conformal Field Theories (RCFT), where only a finite number of representations appears, provides the simplest situation. In this case, it has been possible to study modular covariance and duality in full generality [4]. We give a brief summary of these results in section 1.2. A possible method of classification of RCFT's has emerged from this study. Sections 2 and 3 are devoted to a very elementary illustration of these ideas.

Here, we start from the fusion rules of the theory and try to reconstruct as much as possible of it from these data. The unexpected relationship between the fusion rules of a RCFT and the modular properties of genus-one characters $tr(q^{L_0})$ discovered by Verlinde [9] will be our basic tool and is recalled in section 2.1. We shall use it in section 2.2 to compute the dimensions and central charge of the theory in some particular examples. Finally, section 3 contains the classification of the genus-one partition functions of these theories. This work has been motivated by our attempt to get familiar with Verlinde's work and by a recent work of C. Itzykson on $A_N^{(1)}$ level 1 theories [10]. He formulated a conjecture on the possible modular invariants which we prove in section 3: the modular invariant partition functions are indexed by divisors of N when N is odd and divisors of $N/2$ when N is even.

1 Some concepts in Conformal Field Theory

This section is devoted to a brief introduction to Conformal Field Theory for non specialists. Our presentation is inspired by Segal [8] and relies on the operator formalism [12] [11]. The interested reader can find a very clear presentation in [13]. A more practical point of view is exposed in [14] and [15]. We also describe recent results obtained by Moore and Seiberg in the case of rational conformal field theories.

1.1 Conformal Field Theory

We shall firstly recall what is generally meant by Quantum Field Theory (QFT). We start with a manifold (with a certain structure on it: metric,...) M with boundary ∂M. On this manifold exist some fields which we shall generically denote by φ; these are functions on M with values in a space X (target space). We assume that we are given a local functional $S[\varphi]$ of the fields called the action. Locality means that if we glue two different manifolds M_1 and M_2 together by their common boundaries to get M, we have $S_M[\varphi] = S_{M_1}[\varphi_1] + S_{M_2}[\varphi_2]$. The object we are interested in, is the functional

$$Z[M, f] = \int_{\varphi=f \ on \ \partial M} D[\varphi] \exp\left(-S_M[\varphi]\right).$$

A very well-known example is provided by quantum mechanics where $M = [0, T]$ and $X = \mathbb{R}^3$ (the ordinary space), S_M being the classical action for a particle moving in this space (with or without potential). Then, the functional integral defined above is the probability amplitude for the particle to go from one point to another point in time T. The set of boundary conditions of the fields on ∂M defines the Hilbert space $H_{\partial M}$ of states associated with the manifold ∂M. We are thus able to associate with M a state in $H_{\partial M}$: $|M>$ such that $Z[M,f] = <f|M>$ (scalar product) where $|f>$ is the state associated with the boundary condition f. Note that $Z[M,f]$ can be considered as the wave function associated with $|M>$. This assignment of a state to a (possibly decorated) manifold really defines the field theory.

We are now ready to define a Conformal Field Theory. The precise definition has been given by Segal [8] and here, we shall only outline his definition. A conformal field theory is defined by the assignment of a state to any compact connected Riemann surface M possibly with boundary. To be precise, the state is associated with the surface together with a metric but it is supposed to depend only on the conformal class of the metric [2]. The boundary of such a Riemann surface is a disjoint union of n circles and Segal assumes that $H_{\partial M} = H^{\otimes n}$ where H is the Hilbert space associated with a circle. In the case of a surface with no boundary, the Hilbert space is simply $\mathbf{C}$.

Then, we need "sewing axioms" which express the locality of the theory. Let us consider two Riemann surfaces with boundaries and the surface obtained by gluing the two surfaces together along one of the boundary circles. Let $|M_1> \in H^{\otimes n_1}$ and $|M_2> \in H^{\otimes n_2}$ be the states associated with the two surfaces. We suppose for clarity that the circles we consider are associated with the first spaces. Let ϕ be the canonical antilinear bijection from H onto its topological dual deduced from the Hilbert space structure. Then, we have

$$|M> = ((\phi \otimes \mathbf{1} \otimes \ldots \otimes \mathbf{1})|M_1>).|M_2>$$

where the linear form acts on the first Hilbert space H in the tensor product $H^{\otimes n_2}$. The state we find clearly belongs to $H^{\otimes(n_1+n_2-2)}$. From a surface with at least two boundary circles, we can construct a new surface by gluing along two circles. The associated state is obtained by the same kind of contraction process. These sewing axioms enable us to reconstruct the state associated with any Riemann surface from the state associated with the three holed sphere: we only have to partition the surface into three holed spheres. The state we obtain through this process does not depend on the particular partition of the sphere. By contraction of the state associated with a Riemann surface with suitable states, we can compute physical correlation functions. Inserting a given field on the surface is equivalent to cutting a little disk around the insertion point and putting appropriate boundary conditions on the boundary circle that we have created. Hence, any local field corresponds to an operator acting on H.

In the case of a two-dimensional conformal field theory, it is possible to show that the Hilbert space H is a representation (in general reducible) of two copies of the Virasoro algebra Vir. This algebra is infinite dimensional and is generated by the L_n's for $n \in \mathbf{Z}$ and c. The commutation relations are

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n,-m} \\ [L_n, c] &= 0 \end{aligned}$$

---

[2] Up to a normalisation factor: a Weyl transformation of the metric multiplies the state by a number which can be computed [1] [11]. This phenomenon is called the conformal anomaly.

The L_n's are the Laurent coefficients of the analytic component of the energy momentum tensor $T(z)$ (the other copy arises from the antianalytic component $\bar{T}(\bar{z})$). The eigenvalues of the L_0 and $\bar{L}_0$ operator are called the conformal dimensions of the state. The representations we consider are highest weight representations because $L_0 + \bar{L}_0$ represents the energy of the system and we want it to be bounded from below. The reflexion positivity property of statistical mechanics is expressed through unitarity of the theory: H is a Hilbert space and $L_n^\dagger = L_{-n}$. We shall restrict ourselves to that case. This implies that the conformal dimensions of the states are all positive numbers. We impose the unicity of the vacuum: there exists a unique state of conformal dimensions $(0,0)$ called the vacuum and noted $|0>$. In fact, as explained in [16], with each state $|\varphi>$ is associated a field $\varphi(z)$ such that $\varphi(z)|0>=|\varphi>$. The state $|0>$ is therefore related to the identity operator.

1.2 Rational Conformal Field Theories

At present time, there is no way to study a general conformal field theory: the Hilbert space H is in some sense too complicated (infinite sum of irreducible representations of $Vir \otimes \overline{Vir}$). Of course, the greater the symmetry of the theory, the easier its study is. In some cases, there exists an extension A of the Virasoro algebra such that H is a **finite** sum of irreducible representations of $A \otimes \bar{A}$. This is the simplest situation and we call it a rational conformal field theory (RCFT). We shall write the following decomposition:

$$H = \bigoplus_{i,\bar{i}} (V_i \otimes V_{\bar{i}}) \tag{1}$$

where V_i and $V_{\bar{i}}$ are irreducible representations of A. Just as in the case of the Virasoro algebra, A is generated by the Laurent coefficients of holomorphic fields (among which $T(z)$). All the fields in A have integer dimensions (i.e. they are holomorphic fields on $\mathbb{C}^*$) [3]. The states of H arise from the one punctured sphere and the action of a Laurent coefficient of a holomorphic field can be seen as the action of a contour integral of an operator valued differential form. In some sense, we have reduced ourselves to a finite dimensional problem. To illustrate this point, we shall explain how to define the action of the extended algebra in any genus and number of punctures and then, we shall sum up the work of Moore and Seiberg which expresses duality and modular covariance through a finite number of polynomial equations satisfied by finite dimensional matrices.

The action of A generalizes to more complicated Riemann surfaces. We consider a Riemann surface M with n points P_i and a coordinate z_i defined on a disk D_i containing the point P_i. We shall call it a n-punctured surface in the following. Cutting the surface along circles of constant $|z_i|$'s is equivalent to consider a Riemann surface with n boundary circles. Let $Q(z)$ be a holomorphic field of conformal dimension $(h,0)$ and $v_i(z_i)(dz_i)^{1-h}$ a section of the $(1-h)$-th power of $T^{(1,0)}M$ over D_i. Then, we define

$$\rho(Q, M, (v_i)_i) = \sum_i \int_{\partial D_i} Q(z) v_i(z) \frac{dz}{2\pi i} \tag{2}$$

where the contour integral on C_i acts on the Hilbert space associated with ∂D_i. Thus,

[3] And $\bar{A}$ is associated with the antiholomorphic fields.

$\rho(Q, M, (v_i)_i)$ acts on $H^{\otimes n}$ [4]. In the case of a one-punctured sphere, we select a particular Laurent coefficient of $Q(z)$ in this way. In the case of a three holed sphere, one recovers the action of A on $H \otimes H$ defined by Moore and Seiberg in [4]. In the case of the energy momentum tensor, the action of the Virasoro algebra has a geometrical meaning: it represents the change of moduli of the Riemann surface with its punctures and coordinates [12].

Given an irreducible representation V_i of A, we can associate its conjugate representation $V_{\bar{i}}$ such that the chiral field associated with the highest weight state of $V_{\bar{i}}$ has a non-zero two-point function with the chiral field associated with the highest weight state of V_i. Moreover, if we define the genus-one character of V_i by

$$\chi_i(\tau) = tr_{V_i}(q^{L_0 - \frac{c}{24}}), \quad q = \exp(2\pi i \tau), \tag{3}$$

a representation and its conjugate have the same character. If V_0 denotes the representation with highest weight state $|0>$, $V_0 = V_{\bar{0}}$. The matrix $C_i{}^j = \delta_{\bar{i}}{}^j$ is called the charge conjugation matrix.

Conformal blocks.

Let Σ be the Riemann surface with n punctures and coordinates which we consider; we can write the associated vector as

$$|\Sigma> = \sum_{\substack{i_1,\dots,i_n \\ \bar{\imath}_1,\dots,\bar{\imath}_n ; a}} |\Sigma^a_{i_1,\dots,i_n}> \otimes |\Sigma^a_{\bar{\imath}_1,\dots,\bar{\imath}_n}>$$

where

$$|\Sigma^a_{i_1,\dots,i_n}> \in \bigotimes_{j=1,\dots,n} V_{i_j}$$

and the same for the antiholomorphic part. These $|\Sigma^a_{i_1,\dots,i_n}>$ are called the conformal blocks on the Riemann surface Σ. If we contract the vector $|\Sigma>$ with an element of $H^{\otimes n}$, we compute the correlation function of the corresponding fields on Σ. Therefore, this correlation function is of the form

$$<\varphi_1, \cdots, \varphi_n>_\Sigma = \sum_a F^{(a)}(m) \bar{F}^{(a)}(\bar{m})$$

where $m, \bar{m}$ are the moduli of Σ with its punctures. For example, in the case of the four-point function on the sphere, these are the positions of the points on the sphere and the functions $F^{(a)}$ are the conformal blocks of [16].

The action of the chiral algebra A on such a state $|\Sigma^a_{i_1,\dots,i_n}>$ is defined through contour integration of operator valued differential forms. We are thus led to certain identities, called the Ward identities which express that if one can shrink the contour to a point without encountering any singularity, the action of the operator on the state we consider should be zero. The action of the L_{-1} operator should also be associated with translations. These conditions restrict the $(|\Sigma^a_{i_1,\dots,i_n}>)_a$ vectors to belonging to a finite dimensional vector space. In the case of the three-punctured sphere with representations i, j, k, its dimension is noted $N_{i,j,k}$. The $N_{i,j,k}$'s are called the fusion

[4] Due to the conformal anomaly, there is a subtlety in the case of the energy momentum tensor of the theory: $T(z)v_i(z)dz$ is not a one-form.

rules. Physically, they represent the number of independent couplings between the representations i, j, k which are consistent with the action of the chiral algebra. As any punctured Riemann surface can be obtained by gluing three spheres together, the fusion rules are very important data of a RCFT.

Duality and modular covariance.

If we want to find conformal blocks associated with a Riemann surface, we have to choose a decomposition of the surface into three holed spheres sewed together. This determines a ϕ^3 diagram with fixed representations on the external legs. Then, by choosing some representations in the intermediate channels and one particular coupling for each three holed sphere, we determine a conformal block. For a given ϕ^3 diagram, this gives us a basis of the space of conformal blocks associated with the surface we consider (axiom of duality). Changing the topology of the ϕ^3 diagram or gluing the different three spheres together with Dehn twists in the intermediate channels or around the external points provides us with another basis of conformal blocks. Changing the topology of the graph can be realized using braiding or fusing of the fields (see fig 1). The Dehn twists around the external points are just represented by phase factors and the Dehn twists in the intermediate channels can be computed knowing how Dehn twists are represented on the torus. The different bases of conformal blocks that we have introduced at fixed genus and number of punctures can therefore be related by matrices called the duality matrices which can be easily computed from the dimensions of the fields, the central charge of the theory, the braiding and fusing matrices and the Dehn twist matrices in genus-one. Dehn twists around non trivial loops generate the modular group of the Riemann surface with punctures and coordinates at the punctures [6]. These generators verify certain relations and this gives some equations for the different matrices we use (braiding matrices, Dehn twists matrices in genus-one,...). Moreover, starting from a given ϕ^3 graph and performing a particular sequence of braidings and fusings can give back the same ϕ^3 diagram. This gives also rise to particular identities satisfied by the braiding and fusing matrices. Famous examples of these consistency conditions are provided by the Yang-Baxter equation and the pentagonal identity of Moore and Seiberg [4].

In fact, it is possible to prove that a finite number of polynomial equations imply that all consistency conditions arising from the modular groups defining relations and closed cycles in the set of ϕ^3 diagrams with labelled external legs are verified. Henceforth, these equations ensure duality and modular covariance in any genus and with any number of punctures. This completeness theorem has been proved by Moore and Seiberg [4]. As we said before, we are reduced to a finite dimensional problem.

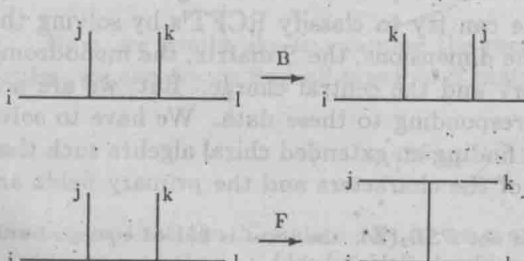

Figure 1: The braiding (B) and fusing (F) operations acting on ϕ^3 diagrams.